Entropy-Based Optimal Sensor Location for Structural Model Updating

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Abstract: A statistical methodology is presented for optimally locating the sensors in a structure for the purpose of extracting from the measured data the most information about the parameters of the model used to represent structural behavior. The methodology can be used in model updating and in damage detection and localization applications. It properly handles the unavoidable uncertainties in the measured data as well as the model uncertainties. The optimality criterion for the sensor locations is based on information entropy, which is a unique measure of the uncertainty in the model parameters. The uncertainty in these parameters is computed by a Bayesian statistical methodology, and then the entropy measure is minimized over the set of possible sensor configurations using a genetic algorithm. The information entropy measure is also extended to handle large uncertainties expected in the pretest nominal model of a structure. In experimental design, the proposed entropy-based measure of uncertainty is also well-suited for making quantitative evaluations and comparisons of the quality of the parameter estimates that can be achieved using sensor configurations with different numbers of sensors in each configuration. Simplified models for a shear building and a truss structure are used to illustrate the methodology.

Key Words: System identification, information entropy, sensors, model updating, structural health monitoring

1. INTRODUCTION

The goal in a structural model updating methodology is to select the model(s) from a parameterized class of models that best fit measured dynamic data according to some criterion. The identified models can then be used for improved structural response predictions or structural damage detection and localization. The quality of the model updating can be judged by the uncertainty in the model parameters and the prediction error. Specifically, the smaller these uncertainties are, the better are the quality of the model updating and the reliability of response predictions or of detection of damage, its localization, and assessment of its severity.

The difficulties associated with the inverse problem of model updating have been addressed by several investigators (e.g., Mottershead and Friswell, 1993; Beck and Katafygiotis, 1998; Katafygiotis and Beck, 1998; Natke and Yao, 1988). The quality of the model updating depends on the class of mathematical models chosen, the measurement error in the data, the number and location of sensors, and the excitation and response bandwidth. Recently, a framework based on a Bayesian statistical methodology has been developed (Beck and Katafygiotis, 1998, Katafygiotis and Beck, 1998; Beck, 1989) to effectively tackle these problems, including the modeling of the uncertainties due to modeling error and measurement noise, the issues of nonuniqueness (Katafygiotis and Beck, 1998; Udwadia and Sharma, 1978), and identifiability (Beck and Katafygiotis, 1998; Katafygiotis, Lam, and Papadimitriou, 1997; Katafygiotis, Papadimitriou, and Lam, 1998), and the problem of reliably computing the response prediction uncertainty. The methodology has also been extended to address issues related to structural damage detection (Beck, Vanik, and Katafygiotis, 1994; Sohn and Law, 1997; Vanik and Beck, 1998; Katafygiotis and Lam, 1997).

This study addresses the problem of improving the quality of the model parameter estimation in relation to the location and number of sensors used. Specifically, the following two issues will be addressed. Given a specified number of sensors, what are the best degrees of freedom (DoF) to instrument in a structure to give the smallest uncertainty when identifying the model parameters using structural response? In addition, what is the improvement in the quality of the parameter estimates as the number of sensors placed in their optimal locations is increased?

Previous work on the subject of optimally locating a given number of sensors on a structure has been carried out by several investigators for both model and modal identification problems (e.g., Shah and Udwadia, 1994; Udwadia, 1994; Kammer, 1991, 1992; Larson, Zimmerman, and Marek, 1994; Penny, Friswell, and Garvey, 1994). In particular, Udwadia (1994) developed a rational statistical-based approach to this problem based on Fisher's information matrix for the model parameters. He proposed that the sensor locations that maximize some norm of the Fisher information matrix be taken as the optimal locations. In his examples, he chose as a "norm" the trace of the matrix. Heredia-Zavoni and Esteva (1998) extended this work to treat the case of large model uncertainties expected in model updating. They proposed that optimal sensor locations should be chosen as the ones that minimize the expected Bayesian loss function involving the trace of the inverse of the Fisher information matrix for each model. Both approaches deal with the first issue of optimally locating a given number of sensors in the structure, and there is no effective approach available for comparing the quality of the parameter estimates as a function of the number of sensors placed in the structure. The answer to the latter issue is useful for making cost-effective decisions regarding structural instrumentation and choice of number of sensors to be placed on the structure.

In the present approach, a different methodology for the optimal sensor location problem is proposed based on the information entropy of the uncertain model parameters (Jaynes, 1978), which is well-suited for addressing both of the aforementioned issues. The Bayesian framework proposed by Beck and Katafygiotis (1998) is extended to the computation of the optimal sensor locations. The uncertainty in these parameters is computed by the Bayesian statistical methodology, and then the information entropy is minimized over the set of possible sensor configurations. Genetic algorithms are well-suited for solving the resulting discrete optimization problem.

It is shown that the results of the present approach are equivalent to those proposed using the Fisher information matrix (Udwadia, 1994), provided the determinant of that matrix is maximized and not its trace. In experimental design, the proposed information entropy is a single measure that can be further used to explore, compare, and evaluate the benefits from placing additional sensors in the structure and the benefits from measuring additional structural modes. This information can help the experimentalist in the decision process of designing a cost-effective experiment to improve the quality of the model predictions. Optimal sensor locations are computed for a linear model of a nine-story shear building and a 29-DoF truss model. Parametric studies illustrate how both the minimum entropy of the parameter uncertainties and the optimal sensor configuration depend on the location of sensors, number of sensors, location of actuators, number and type of contributing modes, and the structural parameterization (substructuring) scheme employed.

2. BAYESIAN STATISTICAL METHODOLOGY FOR MODEL UPDATING

Let $\mathbb{M}$ be a parameterized class of structural models chosen to describe the input-output behavior of a structure. Let $\underline{a}$ denote the model parameters that need to be assigned values from a region $S(\underline{a})$ to choose a particular model $M(\underline{a}) \in \mathbb{M}$. In the statistical model updating methodology, the values of the parameters $\underline{a}$ and their associated uncertainty are updated using test data. The derivation of the uncertainty in the parameters $\underline{a}$ of the parameterized class of structural models chosen to represent structural behavior is presented elsewhere (Beck and Katafygiotis, 1998). A brief summary of their formulation and the main results are presented in the following.

Let $\underline{q}(n; \underline{a}) \epsilon \mathbb{R}^{N_d}$ be the output (e.g., accelerations) from a particular model $M(\underline{a})$ at time $t_n = n\Delta t$ at all N_d DoF of the structural model, where Δt is a prescribed sampling interval. Assume that only N_0 DoF are observed, which are specified by the sensor configuration vector $\underline{\delta} \epsilon \mathbb{R}^{N_d}$ with element $\delta_i = 1$ if DoF i is observed, otherwise $\delta_i = 0$. Thus, only N_0 of the δ_i's are nonzero. The system output at N discrete times, $\underline{y}(n) \epsilon \mathbb{R}^{N_0}$, $i = 1 \cdots, N$, at the observed degrees of freedom is

$$\underline{y}(n) = S_0 \underline{q}(n; \underline{a}) + S_0 \underline{e}(n; \underline{a}), \tag{1}$$

where the prediction error $\underline{e}(n; \underline{a})$ is introduced to account for the modeling error and also the measurement error if $\underline{y}$ is the measured system output. The selection matrix $S_0 \epsilon \mathbb{R}^{N_0 \times N_d}$ has only one nonzero element (unity) in each row and no more than one nonzero element in each column. The position of the nonzero elements in the selection matrix S_0 depends only on the sensor configuration vector $\underline{\delta}$.

The uncertainties in the values of the parameters $\underline{a}$ and the prediction error $\underline{e}(n; \underline{a})$ are quantified using probability models. The uncertainty in $\underline{a}$ is described by a probability density function (PDF), which can be obtained using the class of structural models $\mathbb{M}$, the class of probabilistic models for the prediction error $\underline{e}(n; \underline{a})$, and the observed dynamic data $\mathbb{D}$. Using a linear class of models, and modeling the uncertainty in the components of $\underline{e}(n; \underline{a})$ by an independent Gaussian PDF with mean zero and variance σ^2, the updated PDF for the model parameters $\underline{a}$ for the class of models $\mathbb{M}$ is given by the asymptotic expression for large N:

$$p(\underline{a}|\underline{\delta}, \mathbb{D}_N, \mathbb{M}) \cong \frac{[\det Q(\underline{\delta}, \hat{\underline{a}})]^{\frac{1}{2}}}{(2\pi\hat{\sigma}^2)^{\frac{1}{2}N_a}} \exp\left[-\frac{1}{2\hat{\sigma}^2}(\underline{a}-\hat{\underline{a}})^T \; Q(\underline{\delta}, \hat{\underline{a}}) \; (\underline{a}-\hat{\underline{a}})\right], \tag{2}$$

where $\mathbb{D}_N$ is the data for the first N discrete times, and the (i, j) element of $Q(\underline{\delta}, \underline{a}) \epsilon \; \mathbb{R}^{N_a \times N_a}$ is given by the equation (Beck and Katafygiotis, 1998):

$$Q_{ij}\;(\underline{\delta}, \underline{a}) \cong \sum_{n=1}^{N} \frac{\partial \underline{q}(n; \underline{a})^T}{\partial a_i} \; S_0^T S_0 \frac{\partial \underline{q}(n; \underline{a})}{\partial a_j}. \tag{3}$$

The dependence of $Q_{ij}\;(\underline{\delta}, \underline{a})$ on the sensor configuration vector is through the matrix $S_0^T S_0$. It is assumed for simplicity that the choice of the N_0 observed DoF gives a globally identifiable model (Katafygiotis and Beck, 1998), that is, there is a unique most probable model $\hat{\underline{a}}$ based on $\mathbb{M}$ and $\mathbb{D}_N$ that is given by the unique global minimum of

$$J(\underline{a}) = \frac{1}{NN_0} \sum_{n=1}^{N} \left\| \underline{y}(n) - S_0 \underline{q}(n; \underline{a}) \right\|^2, \tag{4}$$

and the most probable value of the prediction error parameter $\hat{\sigma}^2 = J(\hat{\underline{a}})$ is assumed small.

A more convenient form for $Q_{ij}\;(\underline{\delta}, \underline{a})$ can be derived that involves explicitly the elements of the sensor configuration vector $\underline{\delta}$. Specifically, let $\sigma f(i)$ be a discrete function that gives the correspondence between the system output y_i of the vector $\underline{y}$ and model response $q_{f(i)}$ of the vector $\underline{q}$, i.e., $f(i) \; \epsilon \; \{1, \ldots, N_d\}, \forall i \; \epsilon \; \{1, \ldots, N_0\}$ and $i \neq k \Rightarrow \sigma f(i) \neq \sigma f(k)$. The selection matrix S_0 can be represented as

$$S_0 = \sum_{i=1}^{N_0} E_{if(i)}\;, \tag{5}$$

where $E_{ik} \; \epsilon \; \mathbb{R}^{N_0 \times N_d}$ has all zero elements except for unity for the element in the ith row and kth column. The matrix $S_0^T S_0$ appearing in (3) can now be written in terms of $\underline{\delta}$ as follows:

$$S_0^T S_0 = \sum_{i=1}^{N_0} \sum_{j=1}^{N_0} E_{if(i)}^T \; E_{jf(j)} \; = \sum_{i=1}^{N_0} E_{f(i)i} \; E_{if(i)} \; = diag(\delta_1, \ldots, \delta_{N_d}) \,. \tag{6}$$

Thus, (3) can be written in a more convenient form:

$$Q_{ij}\;(\underline{\delta}, \underline{a}) \cong \sum_{l=1}^{N_d} \delta_l P_{ij}^{(l)}\;(\underline{a}), \tag{7}$$

where the quantities

$$P_{ij}^{(l)}\;(\underline{a}) = \sum_{n=1}^{N} \left[\frac{\partial q_l(n; \underline{a})}{\partial a_i} \frac{\partial q_l(n; \underline{a})}{\partial a_j} \right] \tag{8}$$

depend only on the chosen model and its response at a particular DoF and are independent of the sensor configuration vector $\underline{\delta}$.

3. OPTIMAL SENSOR LOCATION: CASE OF SMALL MODEL UNCERTAINTIES

The optimal value $\underline{\hat{a}}$ of $\underline{a}$ is simply the most probable $\underline{a}$ based on $\mathbb{D}_N$ and $\mathbb{M}$. When we are doing experimental design for choosing the sensor location, $\mathbb{D}_N$ is not known and so $\underline{\hat{a}}$ is uncertain. Note also that for a given sensor configuration, $Q(\underline{\hat{a}})$ depends on $\mathbb{D}_N$ only through $\underline{\hat{a}}$.

Suppose the input (excitation) history $\hat{Z}_N$ is prescribed for the test to identify the model parameters $\underline{a}$. By the previous assumptions, the uncertainty in the system output history, $\underline{y}_n$, $n = 1, \cdots, N$, is modeled by a Gaussian PDF with mean $S_0\underline{q}(n; \underline{a}_0)$ and variance σ_0^2 for each component, where $\underline{a}_0$ and σ_0^2 are the nominal model parameters and prediction error variance that are chosen by the designer to be representative for the structure and the given classes of models.

By the law of large numbers, as $N \to \infty$:

$$J(\underline{a}_0) = \frac{1}{NN_0} \sum_{n=1}^{N} \left\| \underline{y}(n) - S_0\underline{q}(n; \underline{a}_0) \right\|^2 \to \frac{1}{N_0} \sum_{i=1}^{N_0} \mathsf{E}[\underline{e}_{f(i)}^2] = \sigma_0^2. \tag{9}$$

Also, since $\underline{\hat{a}}$ is also the maximum likelihood estimate, we know that $\underline{\hat{a}} \to \underline{a}_0$ as $N \to \infty$, conditional on the nominal model, so $\hat{\sigma}^2 = J(\underline{\hat{a}}) \to J(\underline{a}_0) = \sigma_0^2$ and

$$Q_{ij}\,(\underline{\delta}, \underline{\hat{a}}) \to Q_{ij}\,(\underline{\delta}, \underline{a}_0) \cong \sum_{l=1}^{N_d} \delta_l P_{ij}^{(l)}(\underline{a}_0). \tag{10}$$

Thus, for large N, the PDF $p(\underline{a}|\underline{\delta}, \mathbb{D}_N, \mathbb{M})$ in (2), which quantifies the uncertainty in $\underline{a}$, is a Gaussian PDF with mean $\underline{\hat{a}} \cong \underline{a}_0$ and covariance matrix $\hat{\sigma}^2 Q(\underline{\delta}, \underline{\hat{a}})^{-1} \cong \sigma_0^2 Q(\underline{\delta}, \underline{a}_0)^{-1}$.

Let $p(\underline{a}|\underline{a}_0, \sigma_0, \underline{\delta}, \mathbb{M})$ denote this Gaussian PDF resulting for large N, which depends on the sensor configuration vector $\underline{\delta}$ and the chosen nominal structural and prediction-error model parameters. We wish to minimize the uncertainty in $\underline{a}$ over the sensor locations, that is, over the δ_i's where exactly N_0 of the δ_i's are unity and the rest are zero. As a measure of the uncertainty in $\underline{a}$, we take its (information) entropy (Jaynes, 1978):

$$\begin{aligned} H_{\underline{a}}(\underline{a}_0, \sigma_0, \underline{\delta}, \mathbb{M}) &= \mathsf{E}_a[-\ln p(\underline{a}|\underline{a}_0, \sigma_0, \underline{\delta}, \mathbb{M})] && (11) \\ &= \frac{1}{2} N_a [\ln(2\pi) + 1 + \ln \sigma_0^2] - \frac{1}{2} \ln \det Q(\underline{\delta}, \underline{a}_0). && (12) \end{aligned}$$

As first shown by Shannon (1949), the entropy is well-known as being a unique measure of probabilistic uncertainty. Note that σ_0^2 does not affect the optimal sensor locations. Thus, minimizing the uncertainty in $\underline{a}$ is equivalent to maximizing the determinant of $Q(\underline{\delta}, \underline{a}_0)$. Note that $Q(\underline{\delta}, \underline{a})$ is always a positive semidefinite symmetric matrix and $Q(\underline{\delta}, \underline{a}_0)$ is positive definite since $\underline{a}$ is globally identifiable. Let $\lambda_i(\underline{\delta}, \underline{a}_0), i = 1, \ldots, N_a$, be the eigenvalues of $Q(\underline{\delta}, \underline{a}_0)$ so that $\lambda_i(\underline{\delta}, \underline{a}_0) > 0$, $\forall i$. The optimal locations for N_0 sensors are given by maximizing

$$\ln \det Q(\underline{\delta}, \underline{a}_0) = \sum_{i=1}^{N_a} \ln \lambda_i(\underline{\delta}, \underline{a}_0), \tag{13}$$

or, equivalently, $\det Q(\underline{\delta}, \underline{a}_0) = \prod_{i=1}^{N_a} \lambda_i(\underline{\delta}, \underline{a}_0)$ over $\underline{\delta} = [\delta_1, \ldots, \delta_{N_d}]^T$.

It is straightforward to ascertain that for N_d DoF and N_0 sensors, the total number of discrete values for the sensor location vector $\underline{\delta}$ is

$$N_s = \begin{pmatrix} N_d \\ N_0 \end{pmatrix} = \frac{N_d!}{N_0!(N_d - N_0)!}. \tag{14}$$

For a sufficiently large number of model degrees of freedom N_d, an exhaustive search over all possible values of $\underline{\delta}$ may be computationally expensive or even prohibitive. Instead, genetic algorithms can be used that are well-suited for this type of discrete optimization problem (Goldberg, 1989; Chan, 1997). A genetic algorithm is used in this work to perform the optimization of the objective function.

The expressions in (7) and (8) are discrete versions of an analogous result derived by Udwadia (1994). They have been derived here without using the result that an efficient unbiased estimator satisfies the Cramer-Rao lower bound. Indeed, $\hat{\underline{a}}$ is simply the most probable $\underline{a}$ based on $\mathbb{D}_N$ and $\mathbb{M}$, whereas in Udwadia's result, Q is evaluated at the "true" value of $\underline{a}$ and $\hat{\underline{a}}$ is the unbiased estimator of $\underline{a}$. Udwadia (1994) maximized the trace $\mathrm{tr}Q(\underline{a}_0) = \sum_{i=1}^{N_a} \lambda_i$. The choice of maximizing the trace, instead of the determinant or any other measure of the Fisher information matrix, was justified due to its computational ease and the efficiency with which the maximization can be carried out. The choice of maximizing $\det(Q)$ or $\ln\det(Q) = \sum_{i=1}^{N_a} \ln \lambda_i$ is justified in the present formulation as giving the smallest amount of uncertainty in the parameters of the structure. It will be demonstrated that the use of the trace in place of the determinant results in sensor configurations that are qualitatively different from the optimal sensor configuration obtained by maximizing $\det(Q)$.

The present formulation of the optimal sensor location problem in terms of the information entropy measure provides a rational procedure for comparing the uncertainty of the estimates of the parameter values between different numbers of sensors placed at their optimal locations in the structure. Specifically, let H be the information entropy for a sensor configuration $\underline{\delta}$, and H_0 be the entropy information for some reference sensor configuration $\underline{\delta}_0$ corresponding to a different number of sensors than those in the configuration vector $\underline{\delta}$. Applying (12) for each case, the change $H - H_0$ of the information entropy corresponding to the change in uncertainty in the values of the parameters for $\underline{\delta}$ and $\underline{\delta}_0$ can readily be obtained in the form

$$H - H_0 = \frac{1}{2} \ln \frac{\det Q(\underline{\delta}_0, \underline{a}_0)}{\det Q(\underline{\delta}, \underline{a}_0)}. \tag{15}$$

Let s^2 be the geometrical mean of the principal variances (eigenvalues) of the covariance matrix $\sigma_0^2 Q(\underline{\delta}, \underline{a}_0)^{-1}$ of the distribution $p(\underline{a}|\underline{a}_0, \sigma_0, \underline{\delta}, \mathbb{M})$. The quantity s^2 can be interpreted as giving the overall spread of the distribution $p(\underline{a}|\underline{a}_0, \sigma_0, \underline{\delta}, \mathbb{M})$ about the mean value of the structural model parameters. For two different distributions corresponding to sensor configuration vectors $\underline{\delta}$ and $\underline{\delta}_0$, it can be readily shown that the parameter-uncertainty ratio

$$\frac{s}{s_0} = \exp\left[\frac{H - H_0}{N_a}\right]. \tag{16}$$

Thus, the ratio of the geometrical means of the standard deviations depends only on the change in information entropy and the number of model parameters involved. Thus, this ratio can be equivalently used as an alternative measure of the change in uncertainty between two cases. In particular, a reduction (or increase) of the entropy corresponds to a reduction (or increase) in the ratio s/s_0, while two sensor configuration cases with the same information entropy correspond to ratio $s/s_0 = 1$. In the applications, the change in uncertainty between cases will be reported in terms of the ratio s/s_0 since the magnitude of this ratio, compared to unity, gives a more direct comparison of the spread of the PDF of $\underline{a}$ about its mean values between two cases than does the magnitude of the entropy change.

4. OPTIMAL SENSOR LOCATION: CASE OF LARGE MODEL UNCERTAINTIES

The aforementioned results assume that the updated values of the model parameters do not deviate significantly from the nominal model parameter values $\underline{a}_0$ and σ_0 chosen during the experimental design so that the updated model is close to the nominal model. However, when an experimental design is being done, the updated model is unknown and thus the best values for σ_0 and $\underline{a}_0$ are uncertain. Consider, for example, the case for which $\underline{a}$ is used to model stiffness terms in a structure and the model updating methodology is used to predict structural damage. Large uncertainty in the values will arise due to the possibility of significant reduction in the stiffness of the structure that may occur due to severe damage. For relatively large uncertainties, equation (12) has to be modified to account for all the possible values of the model parameters $\underline{a}_0$ and the prediction error parameter σ_0 along with the respective plausibility of each possible value.

One way to account for the uncertainty in $\underline{a}_0$ and σ_0 is to explore the sensitivity of Q about the nominal value. Alternatively, the uncertainty in $\underline{a}_0$ and σ_0 can be quantified using a prescribed PDF for $\underline{a}_0$ and σ_0 to represent the designer's uncertainty in the model parameters and the prediction error parameter. In this case, the information entropy measuring the uncertainty in $\underline{a}_0$ and σ_0

$$H_{\underline{a}_0,\sigma_0}(\mathbb{M}) = \mathsf{E}_{a_0,\sigma_0}\left[-\ln p(\underline{a}_0,\sigma_0|\mathbb{M})\right] \tag{17}$$

is prescribed. The problem of optimal sensor location therefore becomes the one of minimizing the expected uncertainty in $\underline{a}$ over all possible values of $\underline{a}_0$ and σ_0 when test data are to be used. Equivalently, the problem can be solved by minimizing the change in uncertainty, that is, the change in information entropy given by

$$\Delta H(\delta) = H_{\underline{a},\underline{a}_0,\sigma_0}(\underline{\delta},\mathbb{M}) - H_{\underline{a}_0,\sigma_0}(\mathbb{M}) \tag{18}$$

$$= \mathsf{E}_{\underline{a},\underline{a}_0,\sigma_0}[-\ln p(\underline{a},\underline{a}_0,\sigma_0|\underline{\delta},\mathbb{M})] - \mathsf{E}_{\underline{a}_0,\sigma_0}[-\ln p(\underline{a}_0,\sigma_0|\mathbb{M})]. \tag{19}$$

Modeling $\underline{a}_0$ and σ_0 as statistically independent and using the relation $p(\underline{a},\underline{a}_0,\sigma_0|\underline{\delta},\mathbb{M}) = p(\underline{a}|\underline{a}_0,\sigma_0,\underline{\delta},\mathbb{M})\,p(\underline{a}_0,\sigma_0|\underline{\delta},\mathbb{M})$, the change in information entropy can be readily simplified in the form

$$\Delta H(\delta) = \int H_{\underline{a}}(\underline{a}_0, \sigma_0, \underline{\delta}, \mathbb{M})\, p(\underline{a}_0)\, p(\sigma_0)\, d\underline{a}_0\, d\sigma_0 \tag{20}$$

$$= \frac{1}{2}N_a \left[\ln(2\pi) + 1 + \int \ln \sigma_0^2\, d\sigma_0\right] - \frac{1}{2}\int \ln \det Q(\delta, \underline{a}_0)\, p(\underline{a}_0)\, d\underline{a}_0. \tag{21}$$

The integral in (20) represents the measure of uncertainty in $\underline{a}$ over all possible values $\underline{a}_0$ and σ_0 weighted by the plausibility of each value of $\underline{a}_0$ and σ_0. This is also equivalent to taking the expectation over $\underline{a}_0$ and σ_0 of the information entropy in (11). Equation (21) is derived from (20) after replacing $H_{\underline{a}}(\underline{a}_0, \sigma_0, \underline{\delta}, \mathbb{M})$ by its equivalent form given in (12) and then simplifying the resulting integral. From equation (21), it is obvious that the uncertainty in σ_0 does not affect the optimal sensor locations since the first term in (21) is independent of the sensor configuration vector $\underline{\delta}$. The optimal sensor locations are obtained by minimizing the change of uncertainty or, equivalently, maximizing the quantity

$$h(\underline{\delta}) = \mathsf{E}_{a_0}\left[\ln \det Q(\underline{\delta}, \underline{a}_0)\right] = \int \ln \det Q(\underline{\delta}, \underline{a}_0)\, p(\underline{a}_0)\, d\underline{a}_0 \tag{22}$$

over $\underline{\delta} = [\delta_1, \ldots, \delta_{N_d}]^T$. The multidimensional numerical integration over $\underline{a}_0$ involved in computing the expectation in (22) can be carried out approximately but efficiently using an asymptotic expansion developed to treat these type of integrals (Papadimitriou, Beck, and Katafygiotis, 1997).

In an alternative formulation (Heredia-Zavoni and Esteva, 1998), the optimal sensor locations are obtained by minimizing the Bayesian loss function given by the expected value of $\text{tr}[Q^{-1}(\underline{\delta}, \underline{a}_0)]$ over $\underline{a}_0$ instead of maximizing the expected value of $\ln[\det Q(\underline{\delta}, \underline{a}_0)]$ over $\underline{a}_0$, as proposed herein. The proposed formulation for the optimal sensor locations yields the best estimates of the parameter values in the sense that these estimates correspond to the minimum uncertainty in the values of the model parameters.

5. APPLICATIONS

5.1. Shear Model of Building

The methodology is applied to a nine-story building represented by the mass-spring model shown in Figure 1. The stiffnesses and masses of the nominal structure are chosen to be equal with $k_0/m_0 = 1450$ for each floor so that the fundamental frequency is 1 Hz. Classical normal modes are assumed, with the modal damping fixed at 5% for all modes.

To study the effects of structural parameterization on the optimal sensor location, results are presented for the following three cases, designated by cases A, B, and C. In case A, the uncertainty in the stiffness is assumed to be fully correlated for all stories, that is, only one parameter a is considered with $k_i = ak_0$, $i = 1, \ldots, 9$. In case B, only three uncertain parameters are considered by dividing the structure into three substructures with $k_1 = k_2 = k_3 = a_1 k_0$, $k_4 = k_5 = k_6 = a_2 k_0$ and $k_7 = k_8 = k_9 = a_3 k_0$. In case C, nine uncertain parameters are considered, one for each story stiffness, so that $k_i = a_i k_0$, $i = 1, \ldots, 9$.

The excitation is assumed to be an impulsive base acceleration of unit magnitude, that is, the base acceleration $a(t) = \delta(t)$, where $\delta(t)$ is the delta function. It should be noted that the

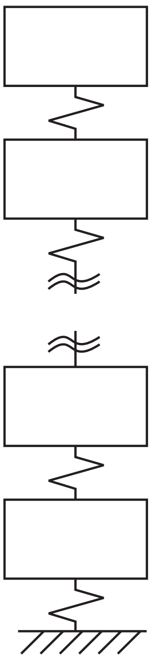

Figure 1. 9-DoF shear building model.

impulse base excitation is chosen to focus on the optimal sensor location for recording seismic response produced by a broadband earthquake excitation. Although, in principle, the impulse excitation will excite all modes of the structure, parametric studies as a function of the number of modes are carried out to study the effect that information from higher modes will have on the sensor locations. Such studies are justified since in real applications the information contained in the measured data from higher modes may be lost due to the larger measurement-noise-to-signal ratio for higher modes, which is partly due to the smaller response excited in the higher modes.

Tables 1 and 2 show the variation of the uncertainty in the parameter $\underline{a}$ as a function of the number of observed modes and number of sensors placed at the optimal locations for cases B and C, respectively. The ratio s/s_0, defined in (16), is used to measure the uncertainty, where the reference value s_0 is chosen to correspond to the optimal sensor configuration case for which all nine degrees of freedom of the structure are instrumented with sensors and all nine modes of the structure are observable. Results for the ratio s/s_0 are presented for increasing numbers of modes where modes are added in the order of increasing natural frequency. For any given number of modes, the value of s/s_0, and thus the uncertainty in the prediction of the value of $\underline{a}$, is seen to reduce as additional sensors are placed in the structure. Similarly, for a given number of sensors, the uncertainty in the value of the model parameters is reduced as additional modes are observable in the response. Increasing the number of sensors and/or the number of modes has an effect of extracting more information from the data, which is reflected in the lower values of s/s_0. If too few sensors and/or modes are used, the model parameters are essentially unidentifiable, which is reflected in the very large values of s/s_0.

Table 1. Information entropy measure s/s_0 for case B.

No. of modes	Number of Sensors 1	2	3	4	5	6	7	8	9
1	3.15×10^6	79.9	51.8	43.0	36.7	32.3	28.9	27.0	26.3
2	14.3	6.9	4.6	3.7	3.2	2.8	2.5	2.4	2.3
3	11.4	5.2	3.7	3.0	2.4	2.1	1.8	1.7	1.6
4	10.7	4.5	3.2	2.6	2.2	1.9	1.6	1.5	1.4
5	10.0	4.2	3.0	2.4	2.0	1.7	1.5	1.3	1.2
6	9.3	4.0	2.9	2.2	1.8	1.6	1.4	1.2	1.1
7	7.9	3.5	2.5	1.9	1.6	1.4	1.2	1.1	1.0
8	7.9	3.5	2.5	1.9	1.6	1.4	1.2	1.1	1.0
9	7.9	3.5	2.5	1.9	1.6	1.4	1.2	1.1	1.0

Table 2. Information entropy measure s/s_0 for case C.

No. of modes	Number of Sensors 1	2	3	4	5	6	7	8	9
1	2.44×10^{12}	6.22×10^{10}	1.85×10^9	6.46×10^7	2.19×10^6	9.04×10^4	4.17×10^3	181.7	152.4
2	1.43×10^9	1.73×10^6	3.56×10^3	104.1	49.5	28.9	22.4	18.9	16.6
3	9.33×10^5	99.6	21.4	11.7	8.8	7.1	6.0	5.3	4.7
4	1.38×10^3	12.1	6.2	4.7	3.7	3.1	2.7	2.4	2.1
5	16.1	6.0	3.8	2.8	2.3	1.9	1.7	1.5	1.4
6	14.1	5.1	3.2	2.4	1.9	1.6	1.4	1.3	1.2
7	12.1	4.3	2.8	2.1	1.7	1.4	1.3	1.1	1.0
8	10.3	4.0	2.6	2.0	1.6	1.4	1.2	1.1	1.0
9	9.4	4.0	2.6	1.9	1.6	1.4	1.2	1.1	1.0

In experimental design, the proposed methodology could be used as a rational procedure for evaluating and weighing the benefits of adding more sensors in the structure against the benefits of exciting and measuring more modes using the existing number of sensors. For example, consider the case for which there are four sensors placed in the structure and four observable modes. The value of the parameter-uncertainty ratio s/s_0 given in Table 2 for this case is 4.70. Exciting and observing one more mode (fifth mode) results in an improved quality of the parameter estimation corresponding to a lower ratio value of 2.79. To achieve approximately the same quality in the estimates of the model parameters using additional sensors with the original four modes, one needs to add at least three more sensors in the structure to get a reduction of the information entropy to the ratio level of 2.68. Thus, the proposed method can help guide the design of the experiment with respect to which direction to proceed in improving the quality of the estimates. The final decision on whether to use more sensors or excite and observe more modes will depend on the number of sensors available and their cost, as well as the feasibility and cost of exciting and measuring more modes.

Figure 2 shows the ratio s/s_0 values as a function of the number of sensors for the case for which all nine modes of the structure are observed in the data. Results of the ratio s/s_0 are presented for both the optimal and worst possible sensor locations. It is clear from these

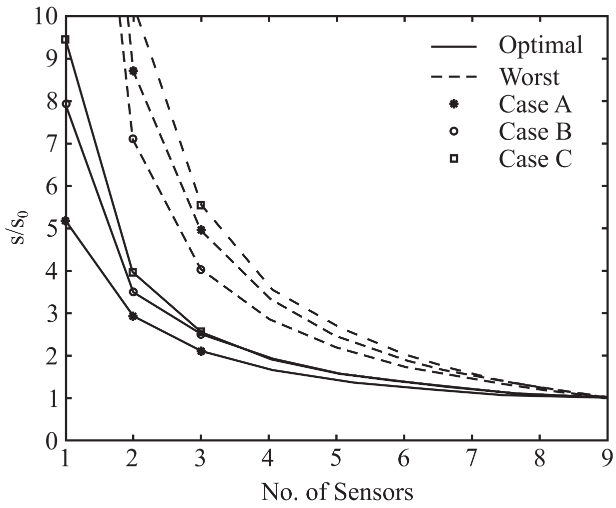

Figure 2. Variation of s/s_0 with number of sensors.

plots that a given number of sensors placed at their optimal configuration may yield estimates of the model parameters that are much better than the ones obtained by a larger number of sensors arbitrarily placed in the structure. For example, two sensors placed at their optimal location for the case A, B, and C will provide a better estimate than, respectively, 4, 3, and 3 sensors placed at the worst location. Thus, optimizing the sensor location in a structure is equivalent to optimizing the cost of instrumentation. It is also seen for each case A, B, and C in Figure 2 that the ratio s_o/s_w between the values of s computed at the optimal and worst sensor locations decreases monotonically with the number of sensors and becomes relatively small as the number of sensors approaches the number of degrees of freedom. These results clearly indicate the importance of optimizing the location of the sensors in the structure, especially for a relatively small number of sensors compared to the number of degrees of freedom.

In case A, the optimal location of N_0 sensors is found to be at the N_0 highest floors of the uniform shear building. The worst possible sensor locations corresponding to the largest uncertainty in the parameters are found to be at the lowest N_0 floors of the shear building. These results are valid independently of the chosen number of contributing modes and the number of sensors.

The optimal sensor locations for cases B and C are given in Tables 3 and 4, respectively. Values of the sensor locations are reported only for the cases for which the problem is identifiable. Unidentifiability can easily be predicted by observing the condition number of the matrix Q. Specifically, one eigenvalue of Q equals zero for an unidentifiable structure. Equivalently, the ratio s/s_0 will be infinite provided that s_0 corresponds to an identifiable case. Usually, due to numerical errors, the eigenvalues are all different from zero and then unidentifiability is predicted by the ratio $|\lambda_{\max}/\lambda_{\min}|$, which is expected to be very large for unidentifiable or ill-conditioned cases. Unidentifiable or almost unidentifiable cases are easily recognized by the very high values, compared to unity, of the ratio s/s_0 as shown in Tables 1 and 2.

Table 3. Optimal sensor locations for case B.

	Number of Sensors							
No. of modes	1	2	3	4	5	6	7	8
1		3 6	3 6 9	3 5 6 9	3 4 6 8 9	3 4 5 6 8 9	2 3 4 5 6 8 9	2 3 4 5 6 7 8 9
2	4	3 9	3 4 9	3 4 8 9	2 3 4 5 9	2 3 4 5 8 9	2 3 4 5 7 8 9	2 3 4 5 6 7 8 9
3	9	3 9	3 4 9	3 4 5 9	2 3 4 8 9	2 3 4 5 8 9	2 3 4 5 6 8 9	2 3 4 5 6 7 8 9
4	3	3 9	3 4 9	3 4 6 9	3 4 5 6 9	2 3 4 6 8 9	2 3 4 5 6 8 9	2 3 4 5 6 7 8 9
5	3	3 9	3 4 9	3 4 5 9	3 4 5 6 9	2 3 4 5 6 9	2 3 4 5 6 8 9	2 3 4 5 6 7 8 9
6	3	3 9	3 4 9	2 3 6 9	2 3 4 6 9	2 3 4 5 6 9	2 3 4 5 6 8 9	2 3 4 5 6 7 8 9
7	3	3 9	3 6 9	2 3 6 9	2 3 5 6 9	2 3 4 5 6 9	2 3 4 5 6 8 9	2 3 4 5 6 7 8 9
8	3	3 9	3 6 9	3 4 6 9	2 3 4 6 9	2 3 4 5 6 9	2 3 4 5 6 8 9	2 3 4 5 6 7 8 9
9	3	3 9	2 3 9	2 3 6 9	2 3 5 6 9	2 3 4 5 6 9	2 3 4 5 6 8 9	2 3 4 5 6 7 8 9

Table 4. Optimal sensor locations for case C.

	Number of Sensors							
No. of modes	1	2	3	4	5	6	7	8
1								1 2 3 4 5 6 7 8
2						1 2 5 6 7 8	1 2 4 5 6 7 8	1 2 4 5 6 7 8 9
3				1 4 5 8	1 2 4 7 8	1 2 3 4 7 8	1 2 3 4 7 8 9	1 2 3 4 5 7 8 9
4		1 9	2 3 9	1 2 3 8	1 2 3 8 9	1 2 3 4 8 9	1 2 3 4 5 8 9	1 2 3 5 6 7 8 9
5	1	1 3	1 2 9	1 2 3 9	1 2 3 4 9	1 2 3 4 7 9	1 2 3 4 5 7 9	1 2 3 4 5 6 7 9
6	1	1 3	1 2 3	1 2 3 9	1 2 3 4 9	1 2 3 4 7 9	1 2 3 4 6 7 9	1 2 3 4 5 6 7 9
7	1	1 3	1 2 3	1 2 3 9	1 2 3 4 9	1 2 3 4 6 9	1 2 3 4 5 6 9	1 2 3 4 5 6 7 9
8	1	1 3	1 2 3	1 2 3 9	1 2 3 4 9	1 2 3 4 6 9	1 2 3 4 5 6 9	1 2 3 4 5 6 7 9
9	1	1 3	1 2 9	1 2 3 9	1 2 3 4 9	1 2 3 4 6 9	1 2 3 4 5 6 9	1 2 3 4 5 6 7 9

Comparing the optimal sensor location results for cases A, B, and C, it is concluded that the optimal location of the sensors depends on the number of sensors placed on the structure, number of modes observed, and the structural parameterization scheme employed. As an example, consider finding the optimal locations of four sensors for cases B and C. The results for case B shown in Table 3 indicate that the exact locations of three of the sensors depend on the number of modes contributing significantly to the response, while one sensor should always be placed at the ninth floor. The results for case C in Table 4 indicate that for five modes or more, three sensors should be placed at the lowest three floors and one sensor should be placed at the highest floor. Note that these locations are different from the ones predicted for case A for which the optimal sensor locations are at the highest four top floors. Note also that the optimal locations for $(N_0 + 1)$ sensors do not always contain the optimal locations for N_0 sensors as a subset.

From the results in Tables 3 and 4, one can conclude in general that for a very small number of sensors, specifically one sensor for case B and up to two sensors for case C, the optimal locations are generally on the lower floors, while for a larger number of sensors, the optimal locations consist of a combination of lower and higher floors. The number of observable modes has some effect on the sensor location problem, but it is not strong.

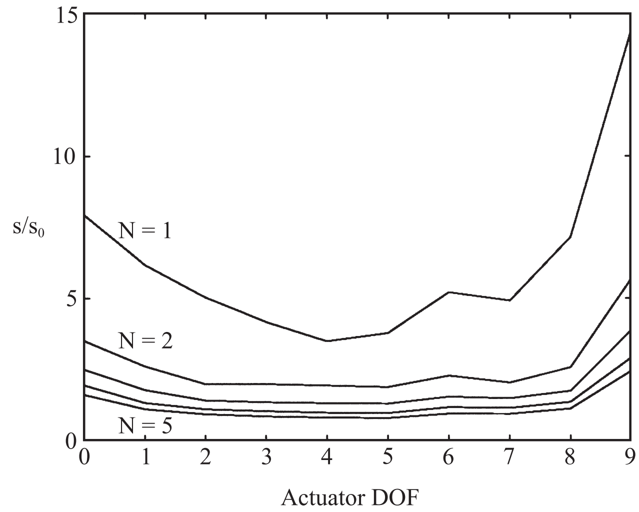

Figure 3. Variation of s/s_0 with actuator position for different numbers, N, of sensors.

Next, the results from the entropy approach are compared to those obtained by maximizing the trace of the Fisher information matrix (Udwadia, 1994). For case A, the results obtained by maximizing $\det(Q)$ or $\mathrm{tr}(Q)$ are identical since Q is a scalar in this case. However, in almost all the results obtained for cases B and C, the optimal sensor location measures $\det(Q)$ and $\mathrm{tr}(Q)$ give qualitatively different results. Specifically, $\mathrm{tr}(Q)$ predicts that for cases B and C, all of the N_0 sensors should be placed at the highest N_0 floors of the shear building, independently of the number of modes and the parameterization used. However, the locations of the sensors predicted by $\det(Q)$ are qualitatively different. In particular, it was found that the optimal sensor locations predicted by $\mathrm{tr}(Q)$ in case C coincide with the worst possible sensor locations predicted by $\det(Q)$. Thus, the measure $\mathrm{tr}(Q)$ gives sensor configurations that may correspond to the highest uncertainty in the values of the parameters $\underline{a}$, so it may give the worst estimates rather than the best.

The effect of the location of the actuator on the uncertainty of the parameter estimates and the optimal sensor location is examined next for case B. Results are presented for the case for which all nine modes are observable. The ratio s/s_0 computed for 1 to 5 sensors placed at their optimal position is plotted in Figure 3 as a function of the location of the actuator providing the impulsive force. The location 0 corresponds to the base excitation while the location i corresponds to the excitation at floor i. It is obvious that in this case the optimal actuator location corresponding to the smallest value of s/s_0 does not depend strongly on the number of sensors placed in the structure. With the exception of the case of one sensor, the optimal actuator location is predicted to be the DoF 5 at midheight of the building. It should be noted that DoF 5 is the optimal actuator location even if 6 to 9 sensors are placed in the structure. It was also found that the optimal actuator location depends strongly on the number of modes that can be excited and observed in the response. Results for the optimal sensor locations as a function of the actuator location are presented in Table 5. It is apparent that the optimal sensor locations depend on the location of the actuator in the structure.

Table 5. Optimal sensor locations for different actuator positions: Case B.

Actuator Position	Number of Sensors 1	2	3	4	5	6	7	8
0	3	3 9	2 3 9	2 3 6 9	2 3 5 6 9	2 3 4 5 6 9	2 3 4 5 6 8 9	2 3 4 5 6 7 8 9
1	3	2 5	2 5 6	2 3 5 6	1 2 3 5 6	1 2 3 4 5 6	1 2 3 4 5 6 7	1 2 3 4 5 6 7 8
2	2	2 5	2 5 6	2 3 5 6	2 3 4 5 6	1 2 3 4 5 6	1 2 3 4 5 6 7	1 2 3 4 5 6 7 8
3	5	2 5	2 3 5	2 3 5 6	1 2 3 5 6	1 2 3 4 5 6	1 2 3 4 5 6 8	1 2 3 4 5 6 7 8
4	4	4 5	4 5 8	2 4 5 8	2 3 4 5 8	2 3 4 5 6 8	2 3 4 5 6 7 8	1 2 3 4 5 6 7 8
5	5	3 5	3 5 7	3 4 5 7	2 3 4 5 8	2 3 4 5 7 8	2 3 4 5 6 7 8	1 2 3 4 5 6 7 8
6	5	4 8	4 5 8	3 4 5 8	3 4 5 7 8	2 3 4 5 7 8	2 3 4 5 6 7 8	1 2 3 4 5 6 7 8
7	7	4 7	3 4 7	3 4 7 8	3 4 5 7 8	3 4 5 6 7 8	1 3 4 5 6 7 8	1 2 3 4 5 6 7 8
8	6	4 7	3 4 7	3 4 6 7	2 3 4 6 7	2 3 4 6 7 8	2 3 4 5 6 7 8	1 2 3 4 5 6 7 8
9	6	3 7	3 6 7	3 4 6 7	3 4 5 6 7	3 4 5 6 7 8	2 3 4 5 6 7 8	1 2 3 4 5 6 7 8

Additional studies on the nine-story building demonstrate that the optimal sensor configuration also depends on the type of the excitation. For example, it depends on whether the excitation is broad-band ambient, impulsive, or forced harmonic. Also, the optimal actuator location depends on the parameterization used. It should be noted that the present methodology could also be useful in experimental design for providing the optimal locations to excite the structure to get the best estimates of the model parameters, given that a fixed number of sensors have already been placed on the structure.

5.2. Truss Structure

The methodology is also applied to a 29-DoF truss structure shown schematically in Figure 4, which can be viewed as a simple bridge model. It is assumed that all members of the truss have the same sizes and that the mass of the structure is uniformly lumped at the nodes of the truss. The values of the members cross-sectional area, A_0, and the lumped mass, m_0, are chosen such that the fundamental frequency of the structure is 0.5 Hz. Classical normal modes are assumed with the modal damping fixed at 5% for all modes. All results correspond to an impulse excitation of unit magnitude along DoF 8 shown in Figure 4. This impulse excitation can be viewed as simulating the excitation from impact hammer tests exciting the structure at its midspan.

To study the effects of structural parameterization on the optimal sensor location, results are presented for the following three cases, designated by cases A, B, and C. In cases A and B, the model parameters to be updated are selected to correspond to the part of the structure close to the midspan. Specifically, the parameters to be updated are the ones corresponding to the stiffness of the members 4, 5, 15, 17, 26, 27, and 16 placed close to the midspan of the structure. This could represent the case for which the stiffnesses at a given location of a structure are to be monitored for sudden changes caused by environmental effects producing damage. In case A, the stiffness is assumed to be fully correlated for members 4, 5, 15, 17, 26, 27, and 16 so that there is only one uncertain parameter to be updated. In case B, the region at the midspan of the bridge is parameterized using four uncertain parameters. Specifically, the following substructuring is considered: $k_4 = k_5 = a_1 k_0$, $k_{15} = k_{17} = a_2 k_0$, $k_{26} = k_{27} = a_3 k_0$, and $k_{16} = a_4 k_0$. In case C, the whole structure is divided into three substructures

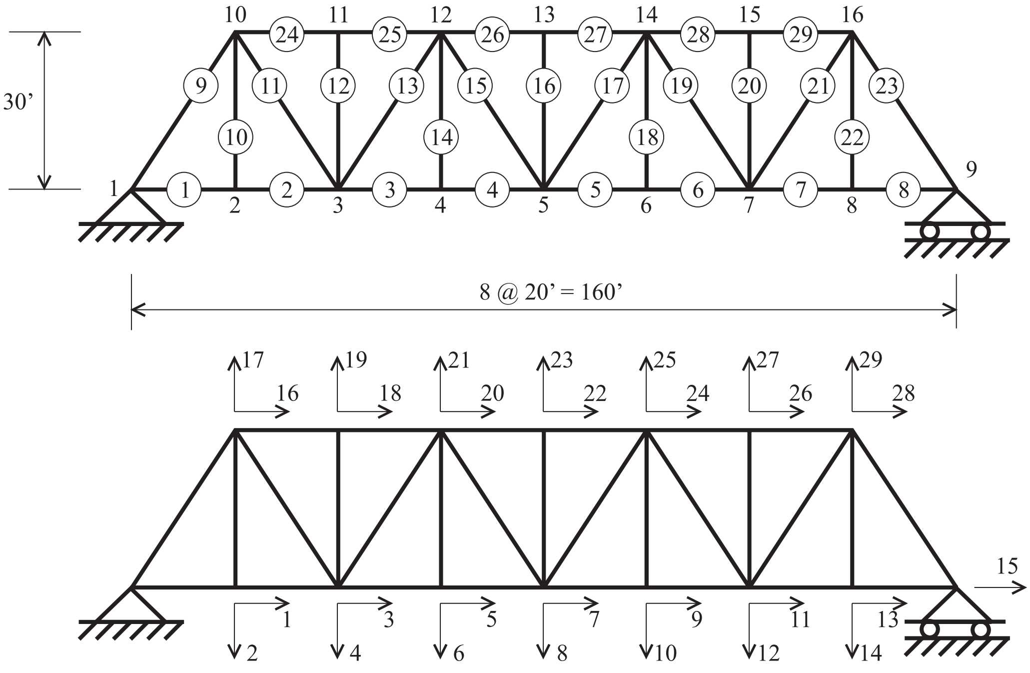

Figure 4. 29-DoF truss structure.

(see Fig. 4), with one parameter corresponding to each substructure, namely, for the first substructure $k_1 = k_2 = k_3 = k_9 = k_{10} = k_{11} = k_{12} = k_{13} = k_{14} = k_{24} = k_{25} = a_1 k_0$, for the second substructure $k_4 = k_5 = k_{15} = k_{17} = k_{26} = k_{27} = k_{16} = a_2 k_0$, and for the third substructure $k_6 = k_7 = k_8 = k_{18} = k_{19} = k_{20} = k_{21} = k_{22} = k_{23} = k_{28} = k_{29} = a_3 k_0$. Case C is representative of a more uniform substructuring suitable for model updating.

Tables 6, 7, and 8 show the variation of the uncertainty in the model parameters $\underline{a}$ as a function of the number of observed modes and number of sensors placed at their optimal locations for cases A, B, and C. Results are presented for maximizing $\det(Q)$ or $\mathrm{tr}(Q)$ but only for the first five modes. The increasing number of modes is produced by adding modes in the order of increasing natural frequency. The ratio s/s_0 is used to measure uncertainty, where for comparison purposes the reference value s_0 is arbitrarily chosen to correspond to the optimal sensor configuration case for which all 29 degreed of freedom of the structure are instrumented with sensors and nine modes of the structure are observable. For any given number of modes, the uncertainty in the value of the model parameters is reduced as additional sensors are placed in the structure. This is expected since increasing the number of sensors has an effect of extracting more information from the data.

To illustrate the use of the entropy values as an aid in experimental design, consider the entropy results shown for case C in Table 8. Assuming first that three modes are observed with one sensor placed at the optimal location, the parameter-uncertainty ratio s/s_0 is 473. Placing a second sensor at the optimal location in the structure will reduce the ratio to the value 227. However, making an effort to excite and observe the fourth mode with only one

Table 6. Information entropy measure s/s_0 for truss: Case A.

	Number of Sensors							
	1		2		3		4	
No. of modes	DET	TR	DET	TR	DET	TR	DET	TR
1	83.3	83.3	42.2	42.2	29.1	29.1	22.3	22.3
2	79.5	79.5	40.3	40.3	28.1	28.1	21.5	21.5
3	79.5	79.5	40.3	40.3	27.9	27.9	21.3	21.3
4	25.7	25.7	15.6	15.6	12.0	12.0	9.9	9.9
5	24.6	24.6	15.2	15.2	11.6	11.6	9.4	9.4

Table 7. Information entropy measure s/s_0 for truss: Case B.

	Number of Sensors							
	1		2		3		4	
No. of modes	DET	TR	DET	TR	DET	TR	DET	TR
1	1.23×10^{11}	3.46×10^{10}	7.12×10^{6}	4.31×10^{6}	2.99×10^{3}	8.04×10^{3}	2.46×10^{3}	6.34×10^{3}
2	1.94×10^{3}	1.94×10^{3}	861.1	1.07×10^{3}	613.7	648.9	480.3	553.6
3	1.68×10^{3}	1.72×10^{3}	576.7	959.4	405.5	483.5	322.6	352.5
4	70.9	146.8	29.6	29.6	20.7	22.2	16.8	18.1
5	55.7	96.0	24.2	24.2	15.6	18.4	12.6	12.9

Table 8. Information entropy measure s/s_0 for truss: Case C.

	Number of Sensors							
	1		2		3		4	
No. of modes	DET	TR	DET	TR	DET	TR	DET	TR
1	4.86×10^{7}	5.35×10^{7}	1.78×10^{3}	1.83×10^{4}	1.23×10^{3}	2.01×10^{3}	927.9	1.49×10^{3}
2	554.7	554.7	263.4	292.2	177.1	213.2	134.2	170.1
3	473.1	530.9	227.2	279.9	151.2	183.9	115.0	141.0
4	51.2	51.2	19.8	25.7	14.0	18.8	11.3	15.0
5	31.1	31.1	16.1	20.3	12.2	14.8	9.8	11.7

sensor will reduce the ratio to the value 51, which is considerably less than 227. Therefore, exciting and observing the fourth mode gives a better quality in the estimated parameters than adding a second, third, or even fourth sensor. Given now that a fourth mode is excited and observed with one sensor, the quality in the prediction can be improved more by adding a second sensor than exciting and observing one additional mode. Similar observations can be made for the other cases in Tables 6, 7, and 8. As it is seen from these tables, in general the answer to the question of whether it is worth adding more sensors in a structure or, instead, make an effort to excite and observe more modes depends on the number of sensors already placed on the structure and the current number of observable modes. Moreover, it depends also on the number and location of the actuators as well as the type of actuator forces. The present formulation is a rational approach for comparing different experimental alternatives and can be useful in the design of a cost-effective experiment.

Table 9. Optimal sensor locations for truss: Case A.

	Number of Sensors							
	1		2		3		4	
No. of modes	DET	TR	DET	TR	DET	TR	DET	TR
1	23	23	8 23	8 23	6 8 23	6 8 23	6 8 21 23	6 8 21 23
2	23	23	8 23	8 23	8 10 23	8 10 23	6 8 10 23	6 8 10 23
3	23	23	8 23	8 23	6 8 23	6 8 23	6 8 10 23	6 8 10 23
4	23	23	8 23	8 23	8 19 23	8 19 23	8 14 19 23	8 14 19 23
5	23	23	8 23	8 23	8 14 23	8 14 23	2 8 14 23	2 8 14 23

Table 10. Optimal sensor locations for truss: Case B.

	Number of Sensors							
	1		2		3		4	
No. of modes	DET	TR	DET	TR	DET	TR	DET	TR
1					8 23 28	6 8 23	8 10 23 28	6 8 21 23
2	23	23	10 23	8 23	10 23 25	8 10 23	10 23 25 28	6 8 10 23
3	10	23	23 27	8 23	10 23 27	6 8 23	10 23 25 27	6 8 10 23
4	8	23	8 23	8 23	8 18 23	8 19 23	8 18 19 23	8 19 23 27
5	8	23	8 23	8 23	8 14 23	8 19 23	8 14 15 23	8 14 19 23

Table 11. Optimal sensor locations for truss: Case C.

	Number of Sensors							
	1		2		3		4	
No. of modes	DET	TR	DET	TR	DET	TR	DET	TR
1			23 27	8 23	8 23 27	6 8 23	8 10 23 27	6 8 21 23
2	23	23	23 24	8 23	8 23 28	8 10 23	8 23 24 28	8 10 23 25
3	27	23	23 27	8 23	23 27 28	8 10 23	19 23 27 28	8 10 23 25
4	23	23	6 23	14 23	6 10 23	2 14 23	6 8 10 23	2 8 14 23
5	23	23	6 23	2 23	6 10 23	2 14 23	6 8 10 23	2 8 14 23

The optimal sensor locations for cases A, B, and C are given in Tables 9, 10, and 11, respectively. Values of the sensor locations in these tables are reported only for the cases for which the problem is identifiable. Results are given for different numbers of modes and different numbers of sensors. In case A, it is seen that the optimal sensor location for one sensor is DoF 23 and the optimal sensor locations for two sensors are DoF 8 and 23, independently of the number of modes contributing to the measured data. The optimal location of an additional third sensor depends on the number of modes that are excited and observed from the data. Similar observation can be made when four sensors are placed in the structure.

Comparing the results for cases A and B, it is observed that the optimal sensor locations depend on the parameterization scheme for the given substructure as well as the number of parameters used. For example, for 4 and 5 modes, the optimal location for one sensor is DoF 8 in case B instead of DoF 23 in case A, while the optimal locations for two sensors are DoF 8 and 23 for both cases A and B. Comparing cases A, B, and C, it becomes evident that the optimal sensor locations depend not only on the parameterization scheme for each substructure but also on the substructuring scheme employed (e.g., localized substructuring as in cases A and B or uniform substructuring over the span of the structure as in case C).

The results from the entropy approach are also compared in Tables 6 to 11 to those obtained by maximizing the trace of the Fisher information matrix. For case A, the results obtained by maximizing $\det(Q)$ or $\mathrm{tr}(Q)$ are identical since Q is a scalar in this case. However, for cases B and C, the measures $\det(Q)$ and $\mathrm{tr}(Q)$ may give different optimal sensor configurations and significantly different parameter uncertainty ratios, depending on the number of modes observed and the number of sensors used. The most significant difference is observed in case B for 4 or 5 modes and one sensor, which is shown in Table 10. Specifically, the optimal sensor locations are DoF 8 and DoF 23 using $\det(Q)$ and $\mathrm{tr}(Q)$, respectively. Although both locations correspond to the vertical degrees of freedom at the midspan of two different nodes, location 23 has significantly higher uncertainty ratio than location 8 as shown in Table 7 for 4 and 5 observable modes. For the other cases in Tables 7 and 8, the entropy measure for the optimal sensor configuration obtained using $\mathrm{tr}(Q)$ is closer to the entropy measure for the optimal sensor configuration obtained using $\det(Q)$.

The results clearly demonstrate that the optimal sensor location proposed by previous approaches based on the Fisher information matrix depends on the norm used. However, the entropy-based measure resolves the issue encountered in previous approaches related to the arbitrariness in selecting an appropriate norm for the Fisher information matrix. Specifically, the norm that best corresponds to the objective of the experiments, which is to minimize the uncertainty in the parameter values, is $\det(Q)$, which arises naturally from the information entropy-based approach. As clearly demonstrated in the numerical studies, the choice of the trace, which is used in other approaches (e.g., Udwadia, 1994; Heredia-Zavoni and Esteva, 1998) primarily because of its convenience in computations, yields different results compared with the entropy-based method, which has a strong information-theoretic basis. Therefore, the use of any norms of the Fisher information matrix other than the determinant is not recommended when determining optimal sensor locations.

Additional studies of the truss model demonstrate that the optimal sensor configuration and the uncertainty measure also depend on the type and location of the excitation.

Finally, some computational aspects of the methodology are briefly discussed. An exhaustive search for the optimal sensor configuration of 1, 2, 3, and 4 sensors requires that the quantity $\ln\det Q(\underline{\delta}, \underline{a}_0)$ involved in (13) be computed for $N_s = 29, 406, 3654$, and 23751 different sensor configurations, respectively, based on (14). It is found that genetic algorithms usually require less than 2% of the total number, N_s, of the function evaluations to obtain the optimal sensor configuration. Thus, genetic algorithms are well-suited to substantially decrease the computational cost, especially for structures with a large number of DoF and for configurations involving a large number of sensors.

6. CONCLUSIONS

The optimal sensor locations are chosen as those that most improve the quality in the estimation of the model parameters in the presence of modeling and measuring errors. The information entropy of the uncertainty of the model parameters is suitable for finding the optimal sensor locations since it is a unique measure of uncertainty. The uncertainty in these parameters is computed using a Bayesian statistical methodology. A genetic algorithm is especially suited for solving the resulting discrete optimization problem involving minimizing the information entropy over all possible sets of sensor configurations. The optimal sensor configuration can be used in conjunction with a system identification technique to provide significantly improved and more reliable estimates of the identified model parameters from test data. The methodology can account for large uncertainties in the model parameters such as, for example, those encountered in damage detection applications where large uncertainties in the location and severity of damage are expected. The optimal sensor locations predicted by the methodology are expected to provide significantly improved estimates of the severity and location of damage. The methodology can be extended to compute the sensor/actuator locations, which provide the best estimates of the modal properties of the structure. The methodology is general and is applicable to a wide range of linear and nonlinear parametric models of structural behavior, including modal-based models with normal and nonnormal modes. Of course, the optimal sensor locations may depend on the class of models that is chosen to represent the structure.

The applicability and features of the optimal sensor location methodology were explored using examples based on linear models of structural behavior. These studies show that the optimal sensor configuration depends on the number of contributing modes, the parameterization scheme employed, and the type and location of excitation. In experimental design, the proposed information entropy is a single measure that can be further used to explore, compare, and evaluate the benefits from placing additional sensors in the structure and the benefits from measuring additional structural modes. This information can be used to guide the design of the experiment so that the best, cost-effective quality in the model parameter estimation is accomplished. The final decision on whether to use more sensors and/or excite and observe more modes will depend on the number of sensors available and their cost, as well as the feasibility and cost of exciting and measuring more modes.

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