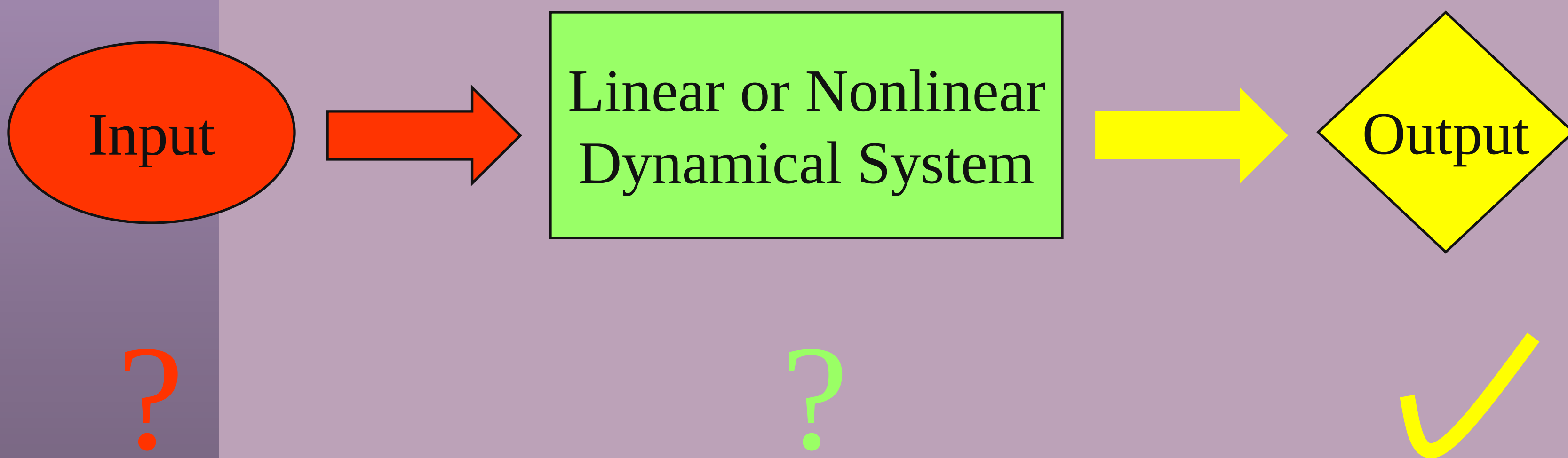
Updating Nonlinear Dynamical Models Using Response Measurements Only

Reference: Yuen and Beck (2003), "Updating properties of nonlinear dynamical systems with uncertain input", ***J. Engng Mech*****. (at website)**

Objective

- Identification of nonlinear dynamical systems using only incomplete noisy response measurements
  - Quantify model and input uncertainties
  - Perform Bayesian updating in frequency domain since FFT of stationary response is nearly Gaussian discrete white noise

System Identification



Model uncertain input by stationary stochastic process

Specification of Model Class

- **Equation of motion** $(N_d$ - DOF model)

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{G}(\mathbf{x}, \dot{\mathbf{x}} \mid \boldsymbol{\theta}_s) = \mathbf{F}(t)$$

$\mathbf{S}_F(\omega|\boldsymbol{\theta}_F)$

- **Observation/Output Equation**

$$\mathbf{y}(n) = \mathbf{q}(n) + \eta(n), \quad n = 0, 1, \ldots$$

$\mathbf{q}$: model response at N_0 $(\leq N_d)$ observed DOF

η: Gaussian prediction error with zero mean and covariance matrix $\boldsymbol{\Sigma}_{n0}$ (max. entropy PDF)

- **Model parameters to be identified**

$$\mathbf{a} = \left\{ \boldsymbol{\theta}_s, \boldsymbol{\theta}_F, \boldsymbol{\Sigma}_{n0} \right\}$$

Bayesian Approach to System ID

- **Bayes Theorem:**

$$\boxed{p(\mathbf{a} \mid \mathbf{Y}_N) = c_1\, p(\mathbf{a})\, p(\mathbf{Y}_N \mid \mathbf{a})}$$

- conditioning on class of models left implicit

- use to update PDF for parameters a based on measured response:

$$\mathbf{Y}_N = [\mathbf{y}(0), \ldots, \mathbf{y}(N-1)]$$

- likelihood difficult to evaluate for nonlinear systems subject to stochastic excitation

Bayesian Spectral Density Approach

- **Key ideas:**
  - **- FFT is nearly (complex) Gaussian white noise for any stationary stochastic process**
  - **- Use likelihood function for spectral density estimates rather than response history**

- **Reference**

  Yuen and Beck (2003), "Updating properties of nonlinear dynamical systems with uncertain input", *J. Engng Mech*. (at website)

Bayesian Spectral Density Approach

- **Illustrate with only one DOF observed:**

$$\mathbf{Y}_N = \left[y(0),\ldots,y(N-1)\right]$$

where $y(n) = q(n) + \eta(n)$

i.e. observed = model + prediction error

- **Spectral Density Estimator from FFT:**

$$\mathbf{S}_{y,N}\left(\omega_k\right) = \frac{\Delta t}{2\pi N}\left|\sum_{n=0}^{N-1}\exp\left(-i\omega_k n\Delta t\right)y(n)\right|^2$$

Bayesian Spectral Density Approach

- **Asymptotic results for $N \rightarrow \infty$:**

- $S_{y,N}(\omega_k)$ is exponentially distributed with mean

$$E\left[S_{y,N}(\omega_k)\right] = E\left[S_{q,N}(\omega_k)\right] + S_{n0}$$

- $S_{y,N}(\omega_k)$ & $S_{y,N}(\omega_l)$ are independent if $k \neq l$

- **Good approximation for large N**

-Estimate mean by simulation or sometimes by

$$E\left[S_{q,N}(\omega_k)\right] = \frac{\Delta t}{2\pi N}\sum_{m=0}^{N-1}[\gamma_m R_q(m\Delta t)]\cos(m\omega_k \Delta t)$$

Bayesian Spectral Density Approach

- **Single set of data using K frequencies**

$$\mathbf{Y}_N = \left[y(0),\ldots,y(N-1)\right] \Rightarrow \mathbf{S}_{y,N}^{K} = [S_{y,N}(\Delta\omega),\ldots,S_{y,N}(K\Delta\omega)]$$

- **Bayes' Theorem:**

$$p(\mathbf{a}\,|\,\mathbf{S}_{y,N}^{K}) = c_2\, p(\mathbf{a})\, p(\mathbf{S}_{y,N}^{K}\,|\,\mathbf{a})$$

$$p(\mathbf{S}_{y,N}^{K}\,|\,\mathbf{a}) \approx \prod_{k=1}^{K} \frac{1}{E[S_{y,N}(\omega_k)\,|\,\mathbf{a}]} \exp\left\{-\frac{S_{y,N}(\omega_k)}{E[S_{y,N}(\omega_k)\,|\,\mathbf{a}]}\right\}$$

Bayesian Spectral Density Approach

- **Multiple non-overlapping sets of data**

$$\mathbf{Y}_N^{(1)},\ldots,\mathbf{Y}_N^{(M)} \Rightarrow \mathbf{S}_{y,N}^{K,(1)},\ldots,\mathbf{S}_{y,N}^{K,(M)}$$

- **Updated PDF for parameters**

$$\boxed{p(\mathbf{a}\,|\,\mathbf{S}_{y,N}^{K,M}) = c_3\, p(\mathbf{a}) \prod_{m=1}^{M} p(\mathbf{S}_{y,N}^{K,(m)}\,|\,\mathbf{a})}$$

Bayesian Approach to System ID

- **Parameter estimation and uncertainty**

  - Optimal parameter $\hat{\mathbf{a}}$ by maximizing $p(\mathbf{a} \mid \mathbf{S}_{y,N}^{K,M})$

  i.e. most probable values given the spectral data

  - $p(\mathbf{a} \mid \mathbf{S}_{y,N}^{K,M})$ locally approximated by a Gaussian PDF with optimal parameters as mean and with covariance matrix:

$$\mathrm{Cov}(\mathbf{a}) = [\mathrm{Hessian}\{-\ln p(\hat{\mathbf{a}} \mid \mathbf{S}_{y,N}^{K,M})\}]^{-1}$$

Example 1: Duffing Oscillator

- **Duffing oscillator**

$$m\ddot{x} + c\dot{x} + kx + \mu x^3 = f(t)$$

- **Gaussian white noise** $f(t): S_f(\omega) = S_{f0}$

- **Good approx. for auto-correlation function from**

$$m\ddot{R}_x(\tau) + c\dot{R}_x(\tau) + (k + 3\mu\sigma_x^2)R_x(\tau) = 0,\ R_x(0) = \sigma_x^2,\ \dot{R}_x(0) = 0$$

- **Expected value of the spectral density estimator**

$$E\left[S_{y,N}(\omega_k)\right] = \frac{\Delta t}{2\pi N} \sum_{m=0}^{N-1} [\gamma_m R_x(m\Delta t)] \cos(m\omega_k \Delta t) + S_{n0}$$

Example 1: Duffing Oscillator

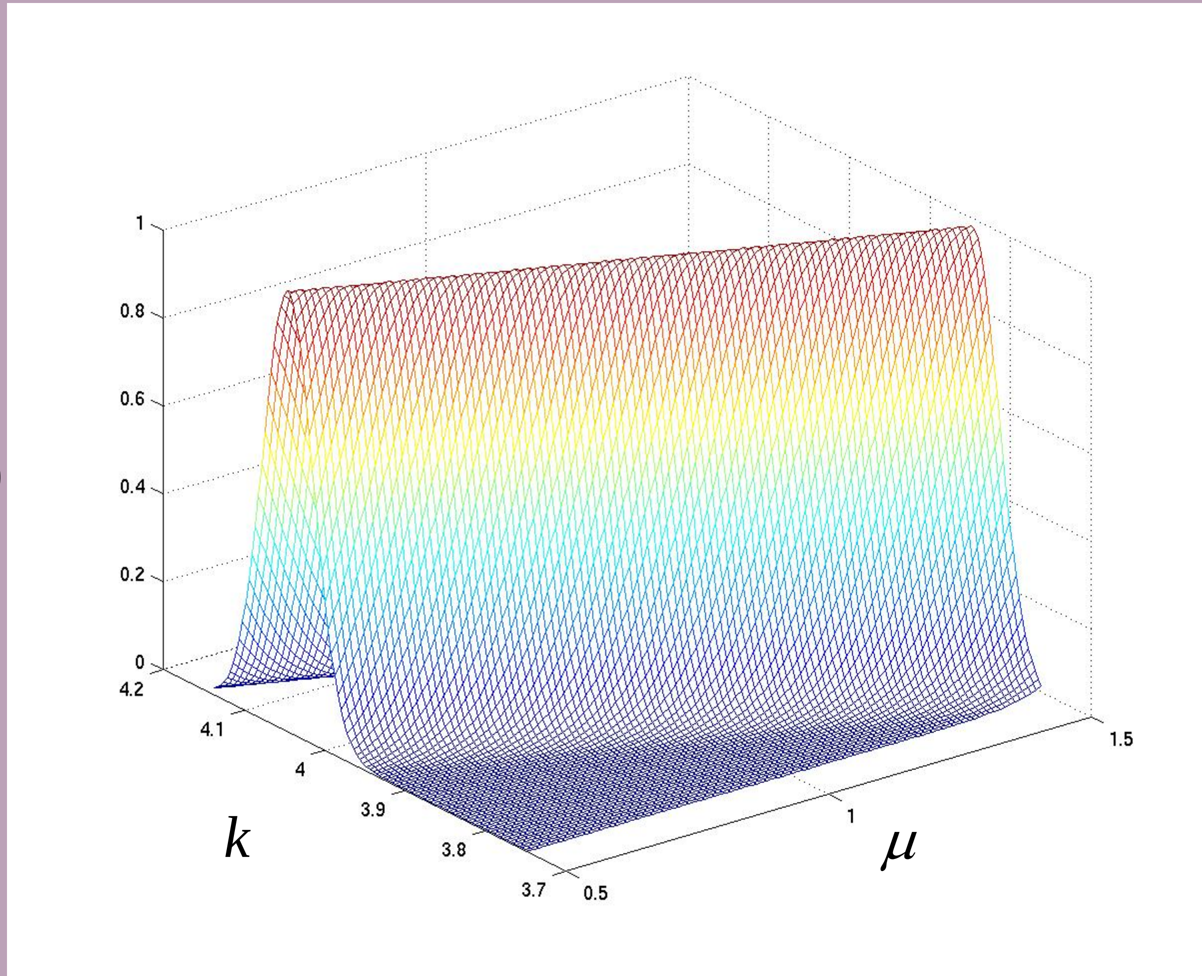
$T = 1000\,\text{sec}, \Delta t = 0.1\,\text{sec},$

$m = 1kg, \text{c} = 0.1\,\text{kg/s}$

$\tilde{k} = 4.0\,\text{N/m}, \tilde{\mu} = 1.0\,\text{N/m}^3$

$\tilde{S}_{f0}^{(1)} = 0.01\,\text{N}^2\text{s}$ and

$\tilde{\sigma}_{n0}^{(1)} = 0.0526\,\text{m}$ (20% noise)

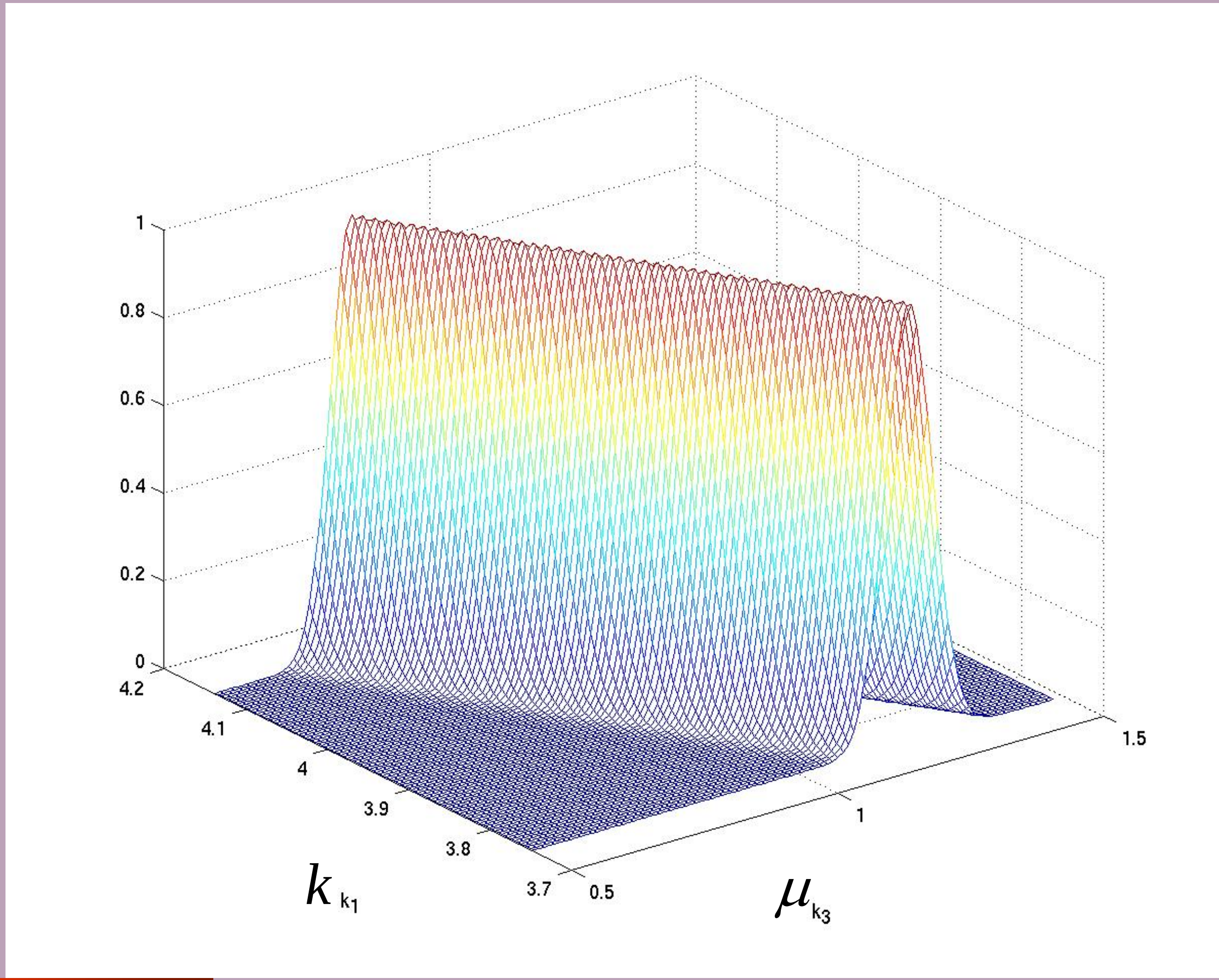


Example 1: Duffing Oscillator

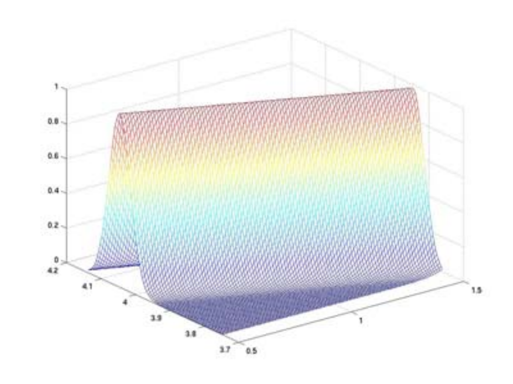
As before but with

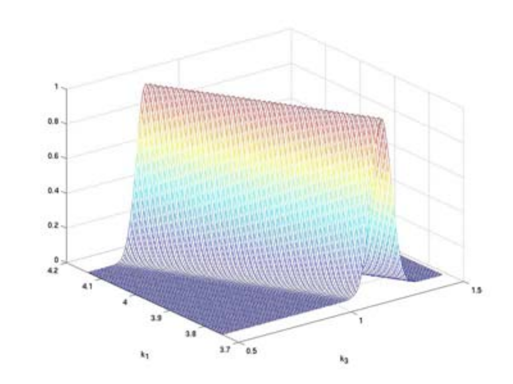
$$\tilde{S}_{f0}^{(1)} = 0.04\,\mathrm{N}^2\mathrm{s} \quad \text{and}$$

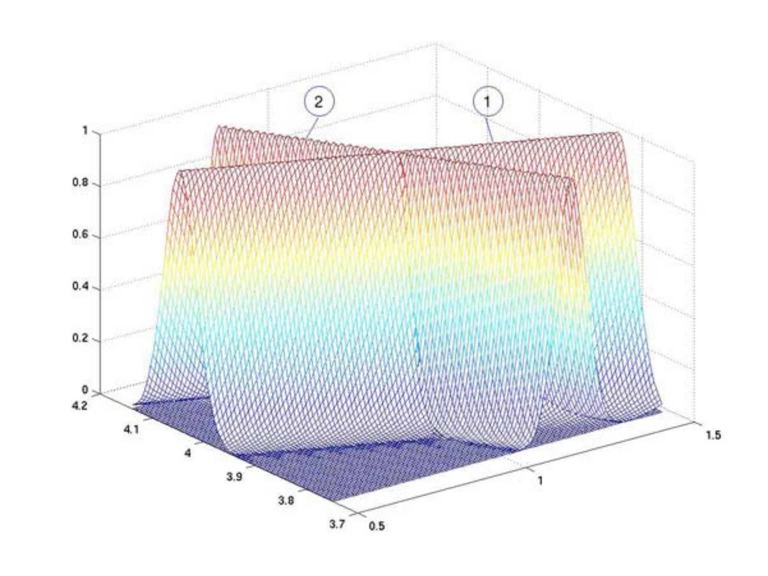
$$\tilde{\sigma}_{n0}^{(1)} = 0.1092\,\mathrm{m}\;(20\%\text{ noise})$$



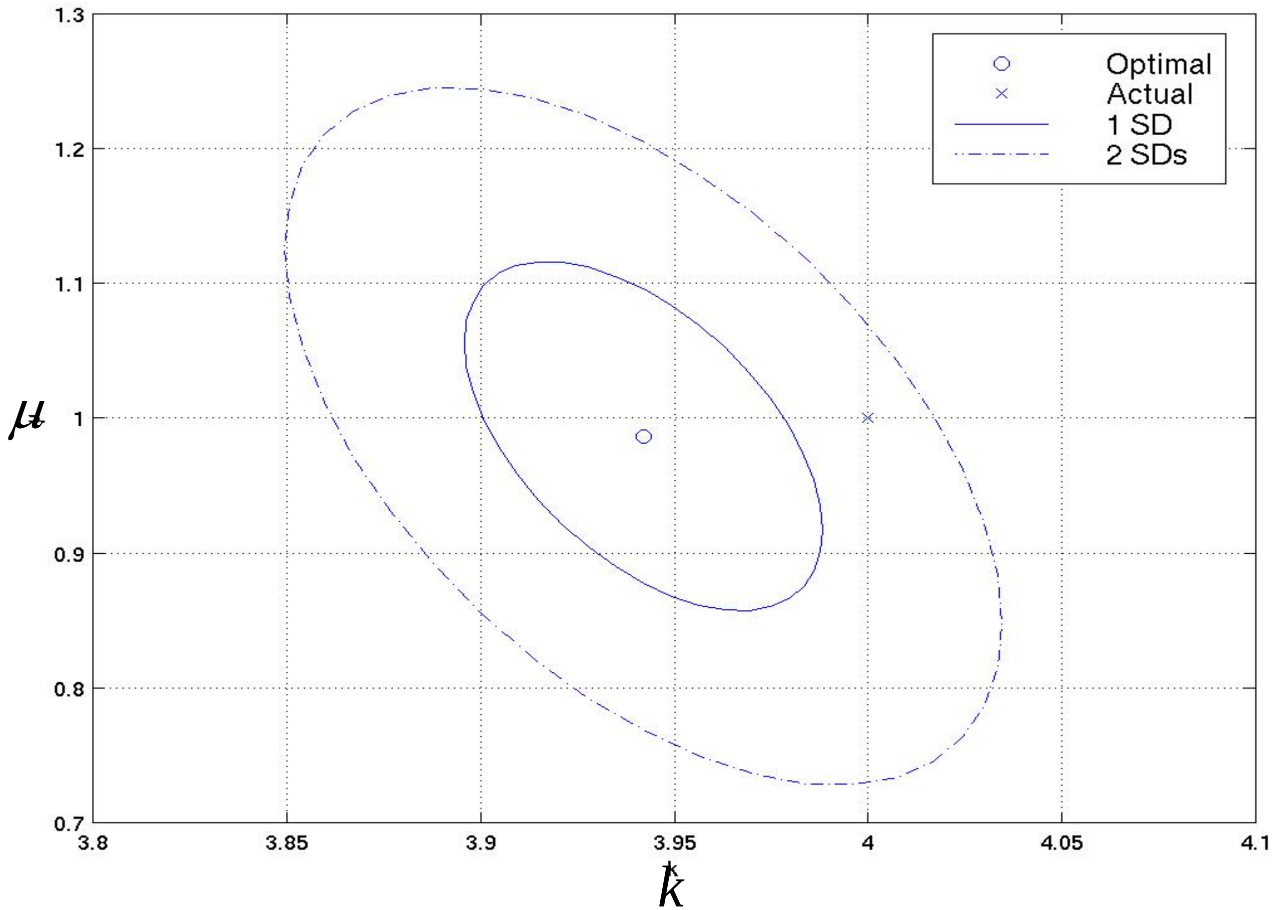
Example 1: Duffing Oscillator



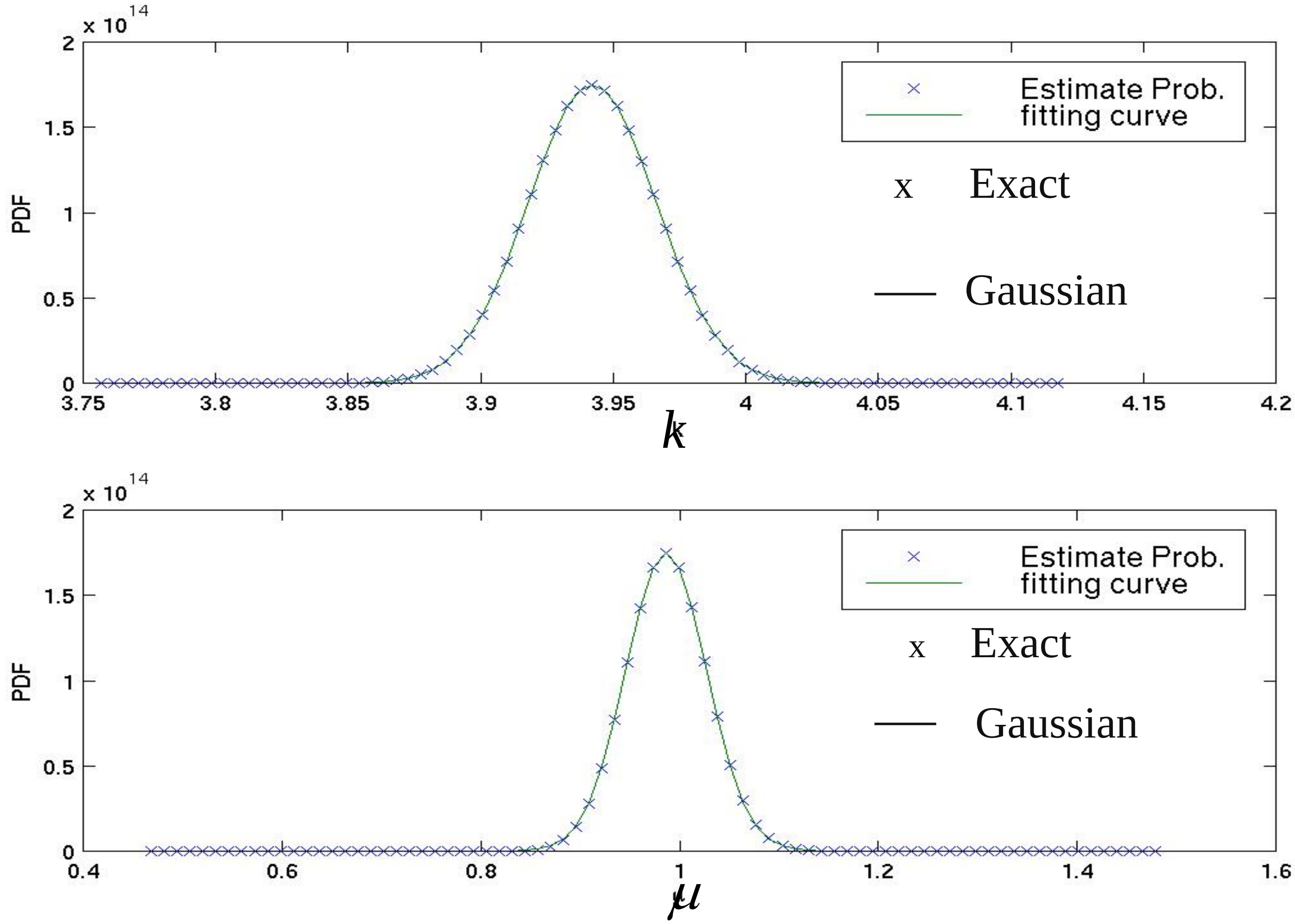




Example 1: Duffing Oscillator



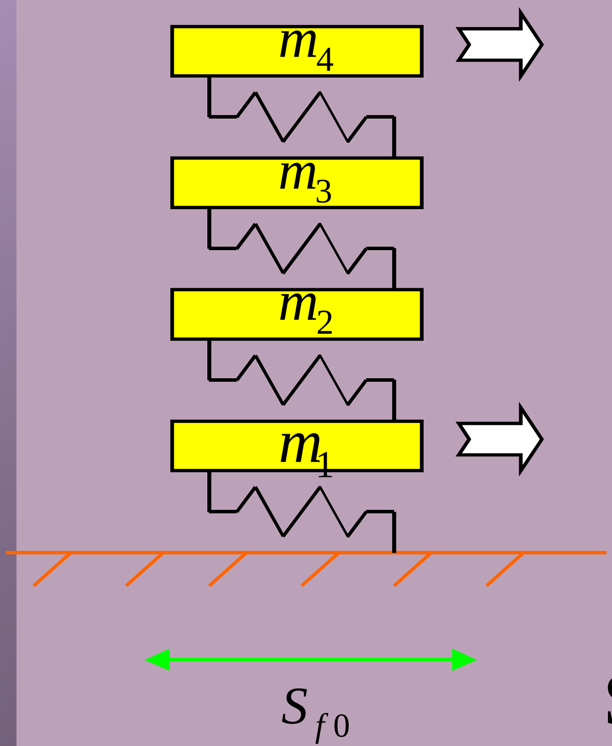
Example 1: Duffing Oscillator



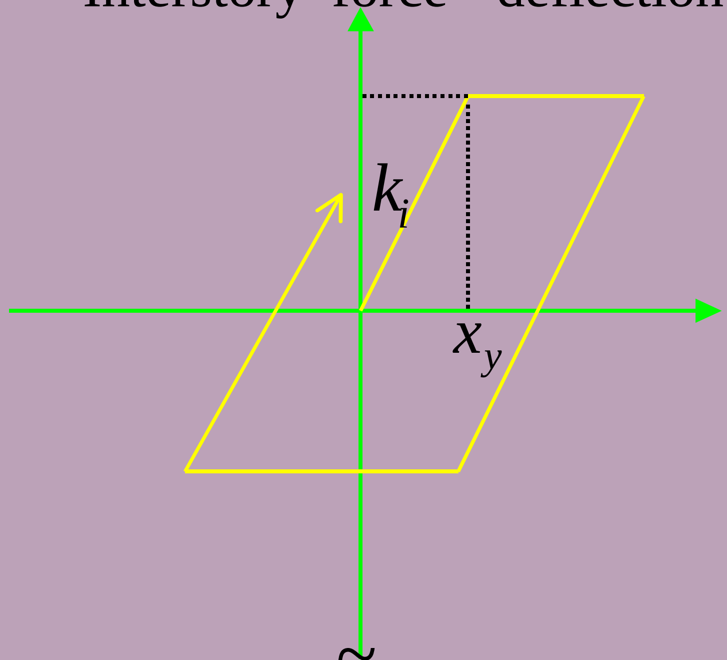
Example 1: Duffing Oscillator

Parameter	Actual	Optimal	S.D.	COV	Error/S.D.
c	0.1000	0.1021	0.0108	0.108	0.20
k	4.0000	3.9420	0.0463	0.012	1.25
μ	1.0000	0.9868	0.1295	0.130	0.10
$S_{f0}^{(1)}$	0.0100	0.0098	0.0005	0.046	0.41
$S_{f0}^{(2)}$	0.0400	0.0454	0.0020	0.051	2.64
$\sigma_{n0}^{(1)}$	0.0526	0.0514	0.0022	0.042	0.55
$\sigma_{n0}^{(2)}$	0.1092	0.1025	0.0045	0.041	1.49

Example 2: 4-DOF Elasto-plastic System

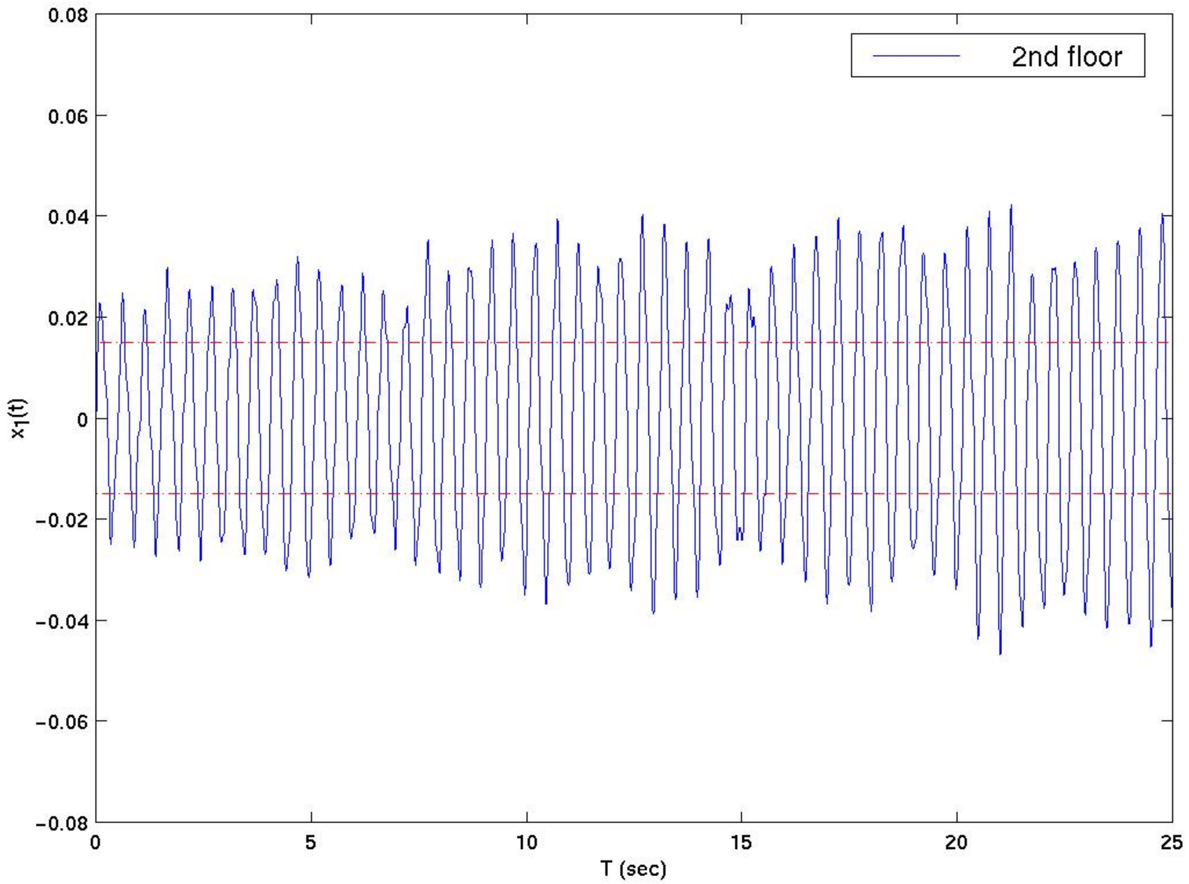


Interstory force - deflection

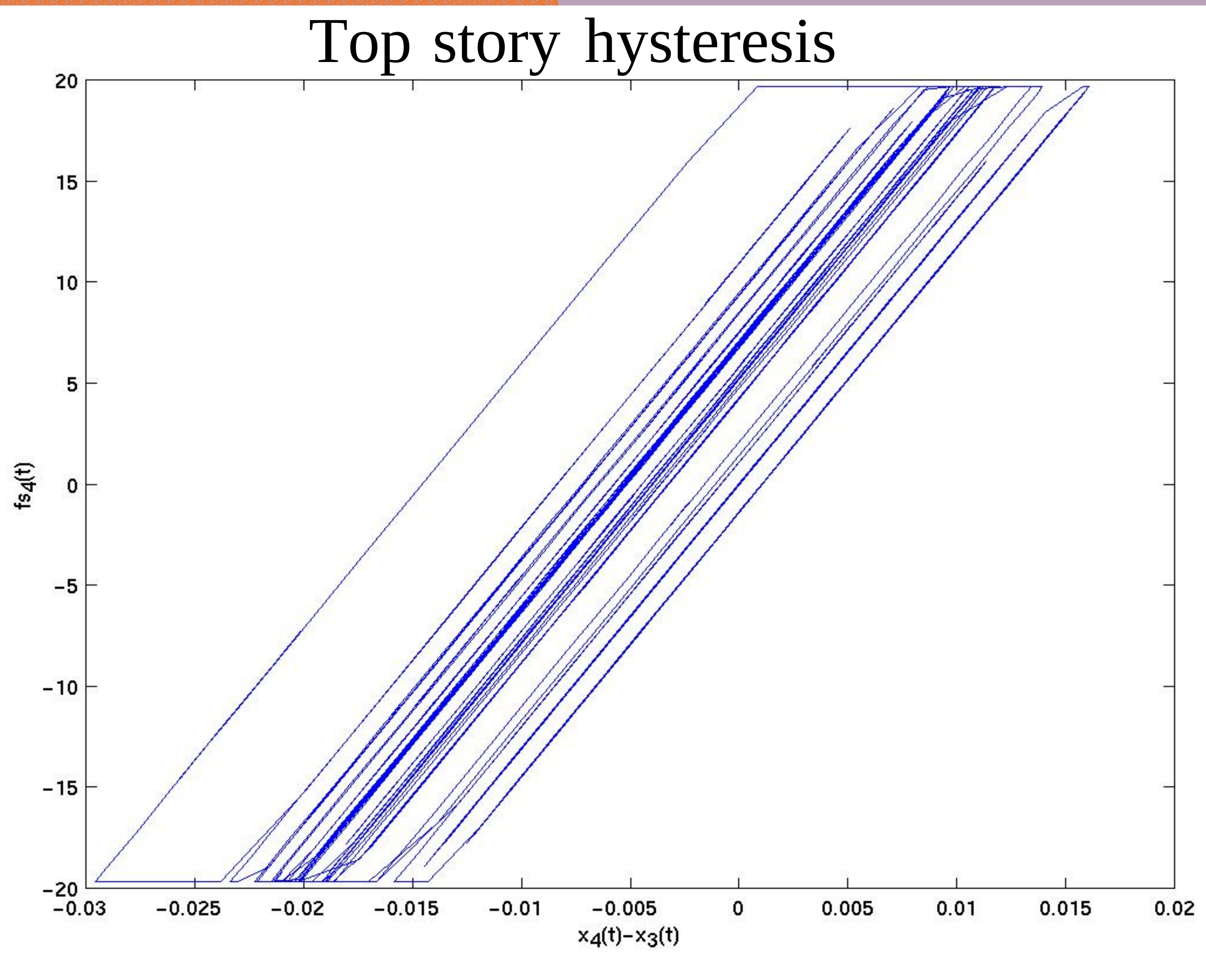


Scale $k_i = \theta_i \tilde{k}_i$ and $x_y = \theta_y \tilde{x}_y$

Example 2: 4-DOF Elasto-plastic System



Example 2: 4-DOF Elasto-plastic System



Example 2: 4-DOF Elasto-plastic System

Parameter	Actual	Optimal	S.D.	COV	Error/S.D.
θ_1	1.0000	1.0122	0.0097	0.010	1.26
θ_2	1.0000	0.9907	0.0089	0.009	1.04
θ_3	1.0000	0.9903	0.0103	0.010	0.95
θ_4	1.0000	0.9947	0.0078	0.008	0.69
θ_y	1.0000	0.9577	0.0533	0.053	0.79
S_{f0}	0.0060	0.0076	0.0008	0.132	2.03
σ_{n1}	0.0022	0.0022	0.0001	0.047	0.03
σ_{n2}	0.0063	0.0062	0.0002	0.040	0.41

Concluding Remarks

- **A Bayesian spectral density approach is available for system identification of nonlinear systems with uncertain input. It is based on the FFT of any stationary response being nearly Gaussian discrete white noise.**

- **The Bayesian probabilistic framework explicitly treats both model uncertainty and excitation uncertainty.**

Concluding Remarks

- The spectral density matrix estimator for a stationary vector process follows a central complex Wishart distribution while it reduces to an exponential distribution for a stationary scalar process.

- The updated (posterior) PDF is usually well approximated by a Gaussian distribution centered at the optimal parameters, so parameter uncertainty may be conveniently and efficiently characterized.

Appendix: Multi-DOF Case

Bayesian Spectral Density Approach

- **Spectral density matrix estimator**

$$\mathbf{S}_{y,N}^{(j,l)}(\omega_k) = \frac{\Delta t}{2\pi N} \sum_{m,p=0}^{N-1} y^{(j)}(m) y^{(l)}(p) \exp\left(-i\omega_k (p-m)\Delta t\right)$$

- **Expected value of the estimator**

$$E[\mathbf{S}_{y,N}(\omega_k)] = E[\mathbf{S}_{q,N}(\omega_k)] + \mathbf{S}_{n,0}$$

$$E\left[S_{q,N}^{(r,s)}(\omega_k)\right] = \frac{\Delta t}{4\pi N} \sum_{m=0}^{N-1} \{[\gamma_m R_q^{(r,s)}(m\Delta t)]\exp(-im\omega_k \Delta t)$$

$$+[\gamma_m R_q^{(s,r)}(m\Delta t)]\exp(im\omega_k \Delta t)\}$$

Bayesian Spectral Density Approach

- **M sets of time histories**

$$\mathbf{Y}_N^{(1)},\ldots,\mathbf{Y}_N^{(N_{set})} \Rightarrow \mathbf{S}_{y,N}^{(1)},\ldots,\mathbf{S}_{y,N}^{(N_{set})}, N_{set} \geq N_o$$

$$\mathbf{S}_{y,N}^{N_{set}} = \frac{1}{N_{set}} \sum_{m=1}^{N_{set}} \mathbf{S}_{y,N}^{(m)}$$

Bayesian Spectral Density Approach

- **Asymptotic results**

1. As $N \to \infty$, $p(\mathbf{S}_{y,N}^{N_{set}}(\omega_k) \mid \mathbf{a})$ asymptotically tends to a central complex Wishart distribution of dimension N_0 with N_{set} DOFs and mean $E[\mathbf{S}_{y,N}(\omega_k) \mid \mathbf{a}]$:

$$p\left(\mathbf{S}_{y,N}^{N_{set}}(\omega_k) \mid \mathbf{a}\right) \approx c_0 \frac{|\mathbf{S}_{y,N}^{N_{set}}(\omega_k)|^{N_{set}-N_o}}{|E[\mathbf{S}_{y,N}^{N_{set}}(\omega_k) \mid \mathbf{a}]|^{N_{set}}}$$

$$\times \exp\left(-tr[E[\mathbf{S}_{y,N}(\omega_k) \mid \mathbf{a}]^{-1} \mathbf{S}_{y,N}^{N_{set}}(\omega_k)]\right)$$

2. $S_{y,N}(\omega_k)$ and $S_{y,N}(\omega_l)$ are independent if $k \neq l$.

Bayesian Spectral Density Approach

- **Finite N**

The above two properties hold approximately

- **Bayes' Theorem**

$$p(\mathbf{a} \mid \mathbf{S}_{y,N}^{N_{set},K}) = c_4\, p(\mathbf{a})\, p(\mathbf{S}_{y,N}^{N_{set},K} \mid \mathbf{a})$$

$$p(\mathbf{S}_{y,N}^{N_{set},K} \mid \mathbf{a}) \approx \prod_{k=1}^{K} p(\mathbf{S}_{y,N}^{N_{set}}(\omega_k) \mid \mathbf{a})$$