Asymptotic Expansions for Reliability and Moments of Uncertain Systems

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Abstract: An asymptotic approximation is developed for evaluating the probability integrals that arise in the determination of the reliability and response moments of uncertain dynamic systems subject to stochastic excitation. The method is applicable when the probabilities of failure or response moments conditional on the system parameters are available, and the effect of the uncertainty in the system parameters is to be investigated. In particular, a simple analytical formula for the probability of failure of the system is derived and compared to some existing approximations, including an asymptotic approximation based on second-order reliability methods. Simple analytical formulas are also derived for the sensitivity of the failure probability and response moments to variations in parameters of interest. Conditions for which the proposed asymptotic expansion is expected to be accurate are presented. Since numerical integration is only computationally feasible for investigating the accuracy of the proposed method for a small number of uncertain system parameters, simulation techniques are also used. A simple importance sampling method is shown to converge much more rapidly than straightforward Monte Carlo simulation. Simple structures subjected to white noise stochastic excitation are used to illustrate the accuracy of the proposed analytical approximation. Results from the computationally efficient perturbation method are also included for comparison. The results show that the asymptotic method gives acceptable approximations, even for systems with relatively large uncertainty, and in most cases, it outperforms the perturbation method.

INTRODUCTION

It is important in reliability based design, and in other engineering problems, to analyze the moments and reliability of structures with uncertain properties subjected to uncertain loads. It is clear, for example, that when a structure is being designed, the environmental loads that the built structure will experience in its lifetime are highly uncertain. Also, response predictions are made during design based on structural models whose parameters are uncertain, because the properties that will be exhibited by the structure when completed are not known precisely.

The uncertain load time history needed in a dynamic analysis of a structure subjected to environmental loads such as earthquakes and wind, is an uncertain-valued function, and so is best modeled by a stochastic process. If the structural parameters are known precisely, then the system reliability and response moments can be calculated using well-known techniques, usually approximate, from random vibration theory. In the more realistic case where the values of the structural parameters are uncertain, these uncertainties are often modeled using a prescribed joint probability density function. The system reliability and response moments are then given by the total probability theorem as particular integrals over all the uncertain parameters. Exact analytical solutions for the reliability and response moments can then be found for only a very limited number of simple systems. Even numerical solutions are limited to cases where there are only a few uncertain parameters, such as the reliability of single degree of freedom systems [e.g., Spencer and Elishakoff (1988)]; otherwise, the computational cost becomes prohibitive. For more realistic systems, Monte Carlo simulation (Rubinstein 1981) can be used to provide more accurate results for both the moments of the response and the system reliability. This method, however, is also computationally very expensive, and often unaffordable, since it requires a very large number of structural analyses to be performed in order to obtain sufficiently accurate results.

First-order and second-order reliability methods (FORM and SORM) have been developed to provide economical computational tools for approximating the structural reliability of uncertain systems when both the system and load uncertainties can be modeled as uncertain-valued variables. The application of the FORM method requires the transformation of the set of variables used to model the uncertainties into the "standard" space of independent normal variables (Madsen et al. 1986). SORM may be formulated either in the space of original variables (Breitung 1991) or in the standard normal space [see, for example, Der Kiureghian et al. (1987)]. For the former formulation, a sound mathematical foundation based on asymptotic analysis was developed by Breitung (1991). FORM or SORM can also be combined with importance sampling techniques to yield accurate estimates of the probability of failure by substantially reducing the number of Monte Carlo simulations required (Schueller and Stix 1987; Bucher 1988).

In recent years, the FORM and SORM methods have also been extended to compute structural reliability for uncertain dynamic systems subjected to stochastic loads such as future earthquakes. They have been applied to the situations where the conditional failure probability for a given set of system parameters is computed from approximations for the first-passage problem from conventional random vibration theory. The methods have been tested for a variety of structural problems, including simple linear and nonlinear systems (Wen and Chen 1987), primary-secondary systems (Igusa and Der Kiureghian 1988), and even hysteretic and geometrically nonlinear multidegree-of-freedom structures (Cherng and Wen 1994). These studies show that the structural uncertainties can have a substantial effect on the reliability of dynamic systems excited by stochastic loads.

Approximate methods for efficiently computing the response moments of dynamic systems have also been developed. The perturbation method, which is computationally the

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Note. Associate Editor: M. P. Singh. Discussion open until May 1, 1998. To extend the closing date one month, a written request must be filed with the ASCE Manager of Journals. The manuscript for this paper was submitted for review and possible publication on February 20, 1996. This paper is part of the ***Journal of Engineering Mechanics***, Vol. 123, No. 12, December, 1997. ©ASCE, ISSN 0733-9399/97/0012-1219–1229/$4.00 + $.50 per page. Paper No. 12677.

least expensive method, works well only for limited cases and for relatively small levels of uncertainties (Koyluoglu 1995). Often it fails to give satisfactory and consistent results, such as in the case of primary-secondary systems, even if the level of uncertainties is small (Singh 1980; Papadimitriou et al. 1995). Alternative approaches (Jensen and Iwan 1992; Papadimitriou et al. 1995) based on expanding the conditional moments of the response for given values of the uncertain system parameters in terms of a series of orthogonal functions of these parameters can be used to overcome these deficiences and provide accurate results. However, these orthogonal expansion methods require excessive computational effort and computer storage for large or even medium-sized systems, and for cases where many uncertain variables are involved. Also, they only apply to a limited class of systems, such as linear systems and nonlinear elastic systems with polynomial type nonlinearities.

In the present study, a new technique based on Laplace's method for asymptotic approximation of integrals is presented for evaluating the type of probability integrals encountered in the analysis of reliabilities or response moments of uncertain systems subject to stochastic excitations. The methodology applies when the conditional reliabilities or conditional expectations of response quantities are available for each value of the uncertain system parameters. It is shown that for computing failure probabilities, the new method is simpler than existing second-order reliability methods developed to treat these types of problems. Simple formulas are also provided for computing the sensitivity of the failure probabilities and moments to system parameters. The proposed method is applied to some simple systems to demonstrate its accuracy. To provide a basis for evaluating this accuracy, accurate numerical solutions for the failure probabilities and response moments are computed by numerical integration and by two simulation methods: straightforward Monte Carlo simulation and an importance sampling technique. The latter proves to be the most efficient way to obtain accurate numerical solutions. On the other hand, the proposed asymptotic method usually gives acceptable accuracy with much less computation.

APPROXIMATIONS FOR CLASS OF PROBABILITY INTEGRALS

Consider the general class of multidimensional integrals of the form

$$I = \int_{\Theta} h(\boldsymbol{\theta})p(\boldsymbol{\theta})\, d\boldsymbol{\theta} \tag{1}$$

where $h(\boldsymbol{\theta})$ and $p(\boldsymbol{\theta})$ = smooth functions for $\boldsymbol{\theta} \in \Theta$; $p(\boldsymbol{\theta})$ = probability density function; and Θ = subregion of $\mathcal{R}^n$. Among other applications, this integral arises in the analysis of the response moments and reliability of uncertain systems subjected to stochastic excitations. Rarely, if ever, can (1) be integrated analytically. Numerical integration can be very costly and is usually unaffordable for more than a few variables. Simulation methods may also require a very large number of integrand evaluations in order to get accurate results. Each integrand evaluation requires $h(\boldsymbol{\theta})$ to be calculated for some $\boldsymbol{\theta}$ value, and this often requires a computationally expensive structural analysis. An asymptotic formula is next presented, which provides an analytical approximation for the integral (1) and which requires only a relatively small number of $h(\boldsymbol{\theta})$ function evaluations.

Asymptotic Approximation

The asymptotic approximation is based on an expansion of the logarithm of the integrand about the point that corresponds to the maximum of the integrand. Consider the case for which the integrand in (1) has a single local maximum $\boldsymbol{\theta}^*$ inside Θ, which is the global maximum over Θ. The idea is to rewrite the integral in the form

$$I = \int_{\Theta} \exp[l(\boldsymbol{\theta})]\, d\boldsymbol{\theta} \tag{2}$$

where

$$l(\boldsymbol{\theta}) = \ln h(\boldsymbol{\theta}) + \ln p(\boldsymbol{\theta}) \tag{3}$$

and expand $l(\boldsymbol{\theta})$ about its maximizing point, which is also $\boldsymbol{\theta}^*$. The following set of equations is satisfied at $\boldsymbol{\theta}^*$:

$$p\frac{\partial h}{\partial \theta_i} + h\frac{\partial p}{\partial \theta_i} = 0, \quad i = 1, \ldots, n \tag{4}$$

provided that $\boldsymbol{\theta}^*$ occurs inside the region Θ, as assumed. However, rather than solving (4), the value of $\boldsymbol{\theta}^*$ is obtained by applying a standard minimization algorithm to $-l(\boldsymbol{\theta})$.

If $l(\boldsymbol{\theta})$ is expanded about $\boldsymbol{\theta}^*$, noting that its derivatives are zero at $\boldsymbol{\theta}^*$, one finds that

$$I = \int_{\Theta} \exp\left[l(\boldsymbol{\theta}^*) - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} L_{ij}(\boldsymbol{\theta}^*)(\theta_i - \theta_i^*)(\theta_j - \theta_j^*) + E(\boldsymbol{\theta})\right] d\boldsymbol{\theta} \tag{5}$$

where $L_{ij}(\boldsymbol{\theta})$, the (i, j) component of the Hessian matrix $\mathbf{L}(\boldsymbol{\theta})$ of $-l(\boldsymbol{\theta})$, is given as

$$L_{ij}(\boldsymbol{\theta}) = -\frac{\partial^2 l(\boldsymbol{\theta})}{\partial\theta_i \partial\theta_j} \tag{6}$$

The integral can be simplified to

$$I = h(\boldsymbol{\theta}^*)p(\boldsymbol{\theta}^*) \int_{\Theta} \exp\left[-\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} L_{ij}(\boldsymbol{\theta}^*)(\theta_i - \theta_i^*)(\theta_j - \theta_j^*)\right] \cdot \exp[E(\boldsymbol{\theta})]\, d\boldsymbol{\theta} \tag{7}$$

By applying Laplace's method of asymptotic expansion to the integral (Bleistein and Handelsman 1986), an asymptotic approximation for I is obtained

$$I(\boldsymbol{\theta}^*) \sim (2\pi)^{n/2} h(\boldsymbol{\theta}^*)p(\boldsymbol{\theta}^*)\frac{1}{\sqrt{\det[\mathbf{L}(\boldsymbol{\theta}^*)]}} \tag{8}$$

which is independent of the expansion error because $E(\boldsymbol{\theta}^*) = 0$. This approximation is valid for $\lambda > 0$, where $\lambda = \min_i\{\lambda_i[\mathbf{L}(\boldsymbol{\theta}^*)]\}$ and $\lambda_i[\mathbf{L}(\boldsymbol{\theta}^*)]$ is the ith eigenvalue of the Hessian matrix $\mathbf{L}(\boldsymbol{\theta})$ evaluated at $\boldsymbol{\theta}^*$. Furthermore, the approximation is asymptotically correct as $\lambda \to \infty$. Specifically, the larger the value of λ, the sharper the peak of the integrand at $\boldsymbol{\theta}^*$ and therefore the more accurate the value of the asymptotic approximation is expected to be.

In the case of a finite number of local maxima in Θ, say $\boldsymbol{\theta}_j^*, j = 1, \ldots, r$, the proposed procedure is modified by simply summing the asymptotic contributions [(8)] computed for each maximum point $\boldsymbol{\theta}_j^*$; that is

$$I = \sum_{j=1}^{r} I(\boldsymbol{\theta}_j^*) \tag{9}$$

This result follows directly from the fact that the integral I can be decomposed into a finite sum of integrals over the disjoint subregions of a partition of Θ, where each subregion contains one and only one maximum point. Of course, some of the contributions in (9) may not be significant.

The computationally most expensive operation in the asymptotic expansion is the search for the maxima points $\boldsymbol{\theta}_j^*$. In some practical applications, only one local maximum exists inside the region Θ, and so it can be readily computed using a local maximization method such as the modified-Newton method. In the case of multiple maxima, more sophisticated optimization methods are required, such as homotopy and relaxation techniques, which are used to reliably obtain most maxima points $\boldsymbol{\theta}_j^*$ (Yang and Beck 1997). The computation of the gradient and Hessian of $-l(\boldsymbol{\theta})$ are required in the aforementioned optimization schemes. These computations could be carried out numerically using finite difference schemes. However, depending on the application, analytical expressions for the gradient and the Hessian of $-l(\boldsymbol{\theta})$ can be developed, thus avoiding possible errors arising from finite difference approximations.

Second-Order Perturbation

The second-order perturbation method offers an approximation of the integral with minimal computational effort. It is based on expanding $h(\boldsymbol{\theta})$ into a Taylor series about the mean $\bar{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}$. Carrying out the expansion and retaining up to second-order terms yields

$$I = h(\bar{\boldsymbol{\theta}}) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} T_{ij}(\boldsymbol{\theta})V_{ij} \tag{10}$$

where

$$T_{ij}(\boldsymbol{\theta}) = \frac{\partial^2 h(\boldsymbol{\theta})}{\partial\theta_i\partial\theta_j} \tag{11}$$

= (i, j) element of Hessian matrix of $h(\boldsymbol{\theta})$; and V_{ij} = elements of covariance matrix of uncertain parameter vector $\boldsymbol{\theta}$ under probability distribution $p(\boldsymbol{\theta})$

$$V_{ij} = E[(\theta_i - \bar{\theta}_i)(\theta_j - \bar{\theta}_j)] \tag{12}$$

Importance Sampling

Importance sampling techniques have been used in reliability analysis (Schueller and Stix 1987; Bucher 1988) to provide accurate estimates of the failure probabilities by substantially reducing the large and possibly prohibitive computer effort required in a straightforward Monte Carlo simulation method. An importance sampling technique is introduced here to efficiently provide an accurate numerical solution that can be used to check the accuracy of the proposed asymptotic method. Comparisons with the straightforward Monte Carlo simulation method are also given to illustrate the advantage of using the importance sampling method.

First, the integral (1) is rewritten in the form

$$I = \int_\Theta \frac{h(\boldsymbol{\theta})p(\boldsymbol{\theta})}{w(\boldsymbol{\theta})} w(\boldsymbol{\theta})\, d\boldsymbol{\theta} = \int_\Theta \kappa(\boldsymbol{\theta})w(\boldsymbol{\theta})\, d\boldsymbol{\theta} \tag{13}$$

where $w(\boldsymbol{\theta})$ = importance sampling density chosen to reduce statistical error of estimate for I. Using simulation on (13), I is estimated by the sample mean of $\kappa = hp/w$

$$I \approx \tilde{I} = \frac{1}{M}\sum_{k=1}^{M} \kappa(\boldsymbol{\theta}^{(k)}) \tag{14}$$

where M = number of simulations and each sample $\boldsymbol{\theta}^{(k)}$ is drawn from importance sampling distribution $w(\boldsymbol{\theta})$. The variance of the estimate $\tilde{I}$ is given by

$$\text{var}[\tilde{I}] = \frac{1}{M(M-1)}\sum_{k=1}^{M} [\kappa(\boldsymbol{\theta}^{(k)}) - \tilde{I}]^2 \tag{15}$$

The variance $\text{var}[\tilde{I}]$ or the coefficient of variation $\text{cov}[\tilde{I}] = \sqrt{\text{var}[\tilde{I}]}/\tilde{I}$ of the estimate $\tilde{I}$ is used to assess the standard error in simulation results, thus providing guidance for terminating the simulation process once the error is below a specified threshold. The choice of $w(\boldsymbol{\theta})$ is a critical factor in obtaining an accurate estimate with fewer simulations than those required in a straightforward Monte Carlo simulation of the original integral (1). The idea is to generate most of the samples in the region that contributes significantly to the integral I so that the importance sampling simulations will converge rapidly to the value of the integral.

Since the main contribution to the integral comes from the domain in the neighborhood of the point $\boldsymbol{\theta}^*$ used in the asymptotic method, it is reasonable to choose $w(\boldsymbol{\theta})$ to have the most probable value equal to $\boldsymbol{\theta}^*$. Also, it can be shown that choosing $w(\boldsymbol{\theta})$ to have the same tail behavior as $p(\boldsymbol{\theta})$ guarantees that the variance of $\kappa = hp/w$ is finite if $h(\boldsymbol{\theta})$ has a finite variance under $p(\boldsymbol{\theta})$. Thus, it seems reasonable to assume that $w(\boldsymbol{\theta})$ has a distribution of the same type as $p(\boldsymbol{\theta})$. Specifically, for a Gaussian $p(\boldsymbol{\theta})$, $w(\boldsymbol{\theta})$ is also chosen to be Gaussian with a most probable value $\boldsymbol{\theta}^*$ and with a covariance matrix the same as that of $p(\boldsymbol{\theta})$. For other distributions for $p(\boldsymbol{\theta})$, there are several ways of applying the importance sampling technique that will guarantee a finite sample variance. One way is to map the original set of variables $\boldsymbol{\theta}$ into a new set of independent Gaussian variables and apply importance sampling to the transformed integral, as just described. Another way, which was used in the present study, is to appropriately choose $w(\boldsymbol{\theta})$ in the original parameter space. Such a choice will depend on the distribution $p(\boldsymbol{\theta})$. In the applications that follow, the components of $\boldsymbol{\theta}$ are assumed to be independently distributed with the ith component θ_i having a lognormal distribution; that is, $p(\theta_i) = (2\pi)^{-1/2}\sigma_p^{-1}\theta_i^{-1} \exp[-1/2(\ln\theta_i - \gamma_p)^2/\sigma_p^2]$. In this case, $w(\theta_i)$ is also chosen to be lognormal with a most probable value θ_i^*, the ith component of $\boldsymbol{\theta}^*$ given by (4), and with $\sigma_w = \sigma_p$ so that $w(\theta_i)$ and $p(\theta_i)$ have the same decay rate for large values of θ_i.

UNCERTAIN DYNAMIC SYSTEMS

The integral (1) arises in the analysis of moments and reliability of uncertain dynamic systems subjected to uncertain excitations. The present work deals with two sources of uncertainties. The first is time-invariant system uncertainties that can be modeled as uncertain-valued parameters. Examples include uncertainties in stiffness, mass, and damping matrices of structural models, as well as member capacities, yield strengths, and so forth. The vector $\boldsymbol{\theta}$ in (1) then consists of these uncertain parameters, while the joint probability density function $p(\boldsymbol{\theta})$ in (1) indicates the relative plausibilities of the possible values of these uncertain parameters in the set Θ. This probability distribution is always conditional on the information used, although this is not explicitly indicated in the notation. If relevant data are available, an updated version of $p(\boldsymbol{\theta})$ can be derived using Bayes' theorem (Box and Tiao 1973). For a large amount of data, an asymptotic approximation of the integral (1) can be made, which relies on the fact that $p(\boldsymbol{\theta})$ is then sharply peaked around certain points in the parameter set Θ (Beck 1989; Beck and Katafygiotis 1991). The asymptotic approximation introduced here can be applied even for small amounts of data, or when $p(\boldsymbol{\theta})$ is chosen subjectively based on engineering experience and other considerations. Of course, the new asymptotic approximation of the integral may not be as accurate in these latter cases as when an updated $p(\boldsymbol{\theta})$ is used based on large samples of data.

The second source of uncertainties is the loading time histories that can be modeled as stochastic processes. The function $h(\boldsymbol{\theta})$ in (1) then represents conditional quantities such as conditional moments or conditional failure probabilities for a given value of $\boldsymbol{\theta}$. Thus, based on the total probability theorem, the integral I represents the total moment or failure probability that accounts for the uncertainties in the system parameters $\boldsymbol{\theta}$, as well as the uncertainties in the loads. Next, methods for approximately computing the probability of failure and the response moments of structures are presented based on approximating the integral (1).

Failure Probability or Reliability

Let $F(\boldsymbol{\theta})$ denote the conditional probability of failure of a structure for a given value of $\boldsymbol{\theta}$. Then the overall failure probability considering the uncertainties in the structural parameters $\boldsymbol{\theta}$ is given in the integral form

$$P_F = \int_{\Theta} F(\boldsymbol{\theta})p(\boldsymbol{\theta})\, d\boldsymbol{\theta} \tag{16}$$

Equivalently, the overall reliability $P_R = 1 - P_F$ can also be written in the integral form

$$P_R = \int_{\Theta} R(\boldsymbol{\theta})p(\boldsymbol{\theta})\, d\boldsymbol{\theta} \tag{17}$$

where $R(\boldsymbol{\theta}) = 1 - F(\boldsymbol{\theta})$ = conditional reliability for a given value of $\boldsymbol{\theta}$; i.e., the conditional probability that the structure will not fail for a given $\boldsymbol{\theta}$. Integrals (16) and (17) are special cases of integral (1) corresponding to the choices of $h(\boldsymbol{\theta}) = F(\boldsymbol{\theta})$ and $h(\boldsymbol{\theta}) = R(\boldsymbol{\theta})$, respectively. Therefore, the approximation developed previously directly applies to the computation of the reliability or failure probability of a structure given the conditional reliabilities or failure probabilities. In the present work, it is assumed that such conditional quantities are available. For example, approximate solutions to the first-passage problem for dynamic systems subjected to stochastic excitations can be used to provide the conditional failure probabilities or conditional reliabilities.

Note that there are many ways of approximating the probability of failure P_F. One way is to directly use the asymptotic approximation (8) with $h(\boldsymbol{\theta})$ replaced by $F(\boldsymbol{\theta})$. Another way is to first approximate P_R using the asymptotic approximation (8) with $h(\boldsymbol{\theta})$ replaced by $R(\boldsymbol{\theta})$ and then compute P_F from $P_F = 1 - P_R$. In fact, there is an infinite number of ways that the integral (1) can be reformulated to obtain approximations of the failure probabilities. Only the accuracy of the asymptotic approximations based on (16) and (17) was examined. It was found that (16) always led to a more accurate estimate of the probability of failure than (17) did, so only the results of the former are presented. The problem with using (17) is that even a very small error in P_R can produce a large relative error in the estimate of P_F because the latter is so small in most problems of interest.

An alternative but less simple way of approximating P_F is to transform the integral (16) to a type of reliability integral over an "unsafe" domain determined by an artificial "limit state" function, and then apply an asymptotic approximation available for this type of integral. This procedure has been carried out by Cherng and Wen (1994) and is outlined in Appendix I. In addition, it is shown in Appendix I that the results available from such an analysis can be greatly simplified, yielding the result obtained by the present asymptotic approximation for P_F. The present procedure, however, is simpler and more direct. In addition, the somewhat arbitrary choice of $p_y(y)$ used in the method of Appendix I is justified by the present procedure as the one that gives a correct asymptotic expansion to approximate the value of P_F.

Response Moments

Denote by $\mathbf{z}$ the state response vector of a structure. It is often desirable to compute the mth moment $E[\eta(\mathbf{z})]$ of the response, where $\eta(\mathbf{z}) = (z_1)^{l_1} \cdots (z_k)^{l_k}$, $l_1 + \cdots + l_k = m$, and operator E denotes mathematical expectation. The computation can be based on the conditional mth moment $E[\eta(\mathbf{z})|\boldsymbol{\theta}]$, which depends on the conditional probability density $p(\mathbf{z}|\boldsymbol{\theta})$ of the response given the system parameters. The response $\mathbf{z}$ could be uncertain because of the uncertain excitation, or because $\boldsymbol{\theta}$ gives only the structural model response that will be different from the actual structural response because of modeling errors (e.g., if $\boldsymbol{\theta}$ specifies a linear model while the actual structure behaves nonlinearly). The latter problem has been studied by Katafygiotis and Beck (1995) for deterministic excitation.

Using the total probability theorem, the unconditional moment $E[\eta(\mathbf{z})]$ is obtained in the form

$$E[\eta(\mathbf{z})] = \int_{\Theta} E[\eta(\mathbf{z})|\boldsymbol{\theta}]p(\boldsymbol{\theta})\, d\boldsymbol{\theta} \tag{18}$$

which is a special case of the general integral (1) with $h(\boldsymbol{\theta}) = E[\eta(\mathbf{z})|\boldsymbol{\theta}]$. Again, available results from random vibration theory can be used to provide solutions for the conditional moments $E[\eta(\mathbf{z})|\boldsymbol{\theta}]$. Exact solutions for $E[\eta(\mathbf{z})|\boldsymbol{\theta}]$ exist for several cases of interest, such as linear systems subjected to external loads modeled by Gaussian, as well as some non-Gaussian, stochastic processes. For nonlinear systems, exact solutions only exist for limited cases. However, available approximations for $E[\eta(\mathbf{z})|\boldsymbol{\theta}]$ developed over the past few decades can be used with the present analysis to compute $E[\eta(\mathbf{z})]$ [e.g., see Soong and Grigoriu (1993); Lin (1967)].

The previous concept can be generalized to compute the expectation of any response quantity of interest from the conditional expectation of the same quantity given $\boldsymbol{\theta}$. Examples for $E[\eta(\mathbf{z})]$ that can be treated include, but are not limited to, the expected stresses, expected total energy, and the expected rate of level exceedance that is useful in simplified structural reliability studies [e.g., Kaspari et al. (1995)].

Sensitivity Analysis

Sensitivity analysis has been mostly used in reliability to identify which uncertainties of the system parameters can significantly influence the failure probability. This concept can also apply to moments or other quantities. The effect of a parameter δ on the integral I can be expressed in terms of a sensitivity function $S(\delta)$, given by

$$S(\delta) = \frac{\text{fractional change in value of } I}{\text{fractional change in system parameter}} = \frac{\delta}{I(\delta)} \frac{\partial I(\delta)}{\partial \delta} \tag{19}$$

which is a measure of the sensitivity of the value of the integral I to a parameter variation, such as a mean or variance involved in the probability density function $p(\boldsymbol{\theta})$. Using (2), $S(\delta)$ can be obtained by

$$S(\delta) = \frac{\delta}{I(\delta)} \int_{\Theta} \frac{\partial l(\boldsymbol{\theta}, \delta)}{\partial \delta} \exp[l(\boldsymbol{\theta}, \delta)]\, d\boldsymbol{\theta} \tag{20}$$

Applying Laplace's asymptotic approximation (Bleistein and Handelsman 1986) for the integral in (20), $S(\delta)$ is approximated by

$$S(\delta) \approx \delta \frac{\partial l(\boldsymbol{\theta}^*, \delta)}{\partial \delta} \tag{21}$$

where $\boldsymbol{\theta}^*$ maximizes $l(\boldsymbol{\theta}, \delta)$. The sensitivity of the failure probability to a parameter δ is a special case of (21) in which $l(\boldsymbol{\theta}, \delta)$ is given by (3) with $h(\boldsymbol{\theta}) = F(\boldsymbol{\theta})$.

For the purpose of illustration, assume that the variable θ_i

is Gaussian and independent of all other variables in $\boldsymbol{\theta}$; then it is straightforward to show that the sensitivities to the mean μ_{θ_i} and the standard deviation σ_{θ_i} of θ_i are given by

$$S(\mu_{\theta_i}) = -\frac{\theta_i^* - \mu_{\theta_i}}{\sigma_{\theta_i}} \frac{\mu_{\theta_i}}{\sigma_{\theta_i}} \quad (22)$$

and

$$S(\sigma_{\theta_i}) = -1 + \left(\frac{\theta_i^* - \mu_{\theta_i}}{\sigma_{\theta_i}}\right)^2 \quad (23)$$

respectively. In particular, as (23) indicates, a measure of sensitivity $S(\sigma_{\theta_i})$ is directly related to the ratio of the distance of the point θ_i^* from the mean μ_{θ_i} to the standard deviation σ_{θ_i}. A similar simple interpretation can be given for $S(\mu_{\theta_i})$. The location of θ_i^* depends on the function $h(\boldsymbol{\theta})$ used in (1).

APPLICATIONS

The accuracy of the asymptotic expansion is investigated by computing the failure probabilities of a single degree of freedom (DOF) linear system shown in Fig. 1(a), and the failure probabilities and second-order response moments of a two DOF linear primary-secondary system shown in Fig. 1(b), when both are subjected to a stationary Gaussian white-noise base excitation with spectral density S. Any response quantity $r(t, \boldsymbol{\theta})$, which is linearly dependent on the components of the state vector describing the response of either of these linear systems, is a Gaussian process since the input is Gaussian. Failure is assumed to occur when the response $r(t, \boldsymbol{\theta})$ reaches some critical level b for the first time. The probability that the stationary portion of duration T of the response $r(t, \boldsymbol{\theta})$ has never reached the value b for the given $\boldsymbol{\theta}$ can be obtained using available results from random vibration theory. These results are based on the expected rate of upcrossing and downcrossing through levels b and $-b$, respectively. For a high threshold level b, it can be assumed that the events of crossing such a level are independent, in which case the conditional failure probability $F(\boldsymbol{\theta})$ is approximated by

$$F(\boldsymbol{\theta}) \approx 1 - \exp[-2\nu(\boldsymbol{\theta})T] \quad (24)$$

where for zero-mean Gaussian processes, the expected rate $\nu(\boldsymbol{\theta})$ of upcrossing through level b for a given $\boldsymbol{\theta}$ is (Soong and Grigoriu 1993)

$$\nu(\boldsymbol{\theta}) = \frac{\sigma_{\dot{r}}(\boldsymbol{\theta})}{2\pi\sigma_r(\boldsymbol{\theta})} \exp\left(-\frac{b^2}{2\sigma_r^2(\boldsymbol{\theta})}\right) \quad (25)$$

where $\sigma_r(\boldsymbol{\theta})$, $\sigma_{\dot{r}}(\boldsymbol{\theta})$ = conditional second moments of response $r(t, \boldsymbol{\theta})$ and its time derivative for a given $\boldsymbol{\theta}$. It is worth mentioning that one can develop analytical expressions for the gradient and the Hessian needed in the proposed asymptotic expansion for evaluating response moments and failure probability.

Uncertain Single Degree-of-Freedom System

Choosing the response $r(t, \boldsymbol{\theta})$ to be the displacement of the mass of the single DOF system with circular frequency ω and damping ratio ζ [Fig. 1(a)], (25) simplifies further to

$$\nu(\boldsymbol{\theta}) = \frac{\omega}{2\pi} \exp\left(-\frac{b^2}{2}\frac{2\zeta\omega^3}{\pi S}\right) \quad (26)$$

The uncertainties in the natural frequency ω, damping ratio ζ, and excitation spectral density S are described by lognormal distributions with most probable values given by $\boldsymbol{\mu}_\theta = \{\mu_\omega, \mu_\zeta, \mu_S\} = \{2\pi, 0.05, 1.0\}$. The threshold level b is chosen to be a multiple of the standard deviation $\sigma_r(\boldsymbol{\mu}_\theta)$ of the response

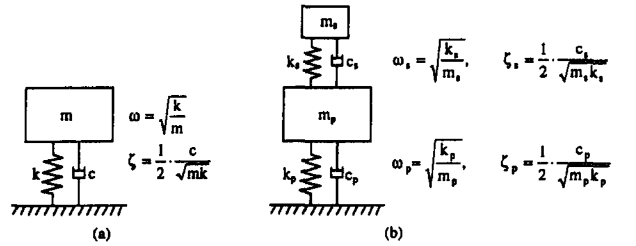

FIG. 1. Systems: (a) Single DOF; (b) Two DOF Primary-Secondary

$r(t, \boldsymbol{\theta})$ computed at the most probable values $\boldsymbol{\mu}_\theta$ of the system parameters, i.e., $b = d\sqrt{2}\sigma_r(\boldsymbol{\mu}_\theta) = d\sqrt{\pi\mu_S/(\mu_\zeta\mu_\omega^3)}$, where d denotes a normalized measure of the threshold level. The duration T of the response is chosen to be $T = 10\ (2\pi/\mu_\omega)$, a multiple of the most probable period of the oscillator. These choices mean that the results for the failure probability P_F are independent of the most probable value assumed for the uncertain system parameters.

At first, the uncertainty in each of the parameters is considered separately in turn while holding the other parameters fixed at their most probable values. The level of uncertainty for a variable θ is measured by the ratio σ_θ/μ_θ, where σ_θ denotes the standard deviation of θ. Figs. 2(a), 2(b), and 2(c) show the failure probabilities as a function of the ratio σ_θ/μ_θ for the three cases of uncertain ω, uncertain ζ, and uncertain S, respectively. On the other hand, Fig. 2(d) shows the failure probabilities as a function of the ratio σ_ω/μ_ω for the case where all three parameters ω, ζ, and S are uncertain. In this case, the standard deviations of the parameters ζ and S are fixed at $\sigma_\zeta = 0.3\mu_\zeta$ and $\sigma_S = 0.2\mu_S$, respectively. All results in Fig. 2 correspond to a normalized threshold level $d = 3$. Table 1 gives the failure probabilities as a function of the normalized threshold level d for the particular case of Fig. 2(d) with $\sigma_\omega/\mu_\omega = 0.2$. For the purpose of comparison, the results obtained from the second-order perturbation method, numerical integration (NI), importance sampling (IS), and straightforward Monte Carlo (MC) simulation are also included in Fig. 2 and Table 1. Only 100 samples are used in Fig. 2 for both the importance sampling technique and the straightforward Monte Carlo simulation.

The following observations can be made regarding the accuracy of the methods. The second-order perturbation method performs well only for the case of one uncertain parameter and small levels of uncertainty. For the case of three uncertain parameters shown in Fig. 2(d), the perturbation method performs poorly even for small levels of uncertainties. In all cases, the asymptotic method performs acceptably well even if the level of uncertainties is relatively large. From Table 1, the accuracy of the perturbation results deteriorates significantly as the threshold level increases. For $d = 4.0$, for example, it underestimates the failure probability by three to four orders of magnitude. Such deterioration is not observed for the asymptotic method.

Fig. 3 shows the location of $\boldsymbol{\theta}^*$ relative to $\boldsymbol{\mu}_\theta$ for the case of Fig. 2(d). Note that $\boldsymbol{\theta}^*$ has moved further away from the most probable value $\boldsymbol{\mu}_\theta$ toward the tails of the distributions assumed for the uncertain variables. It is therefore expected that the importance sampling technique will converge faster than the straightforward Monte Carlo simulations. This is evident in Fig. 2, where the results obtained from the two simulation methods using 100 samples are plotted. The failure probability estimate $\hat{P}_F$ predicted by the two simulation methods as a function of the number of samples is also compared to the failure probability obtained from numerical integration in Fig. 4 for the case of Fig. 2(d) with $\sigma_\omega/\mu_\omega = 0.2$ and for two threshold levels, $d = 3$ in Fig. 4(a) and $d = 4$ in Fig. 4(b).

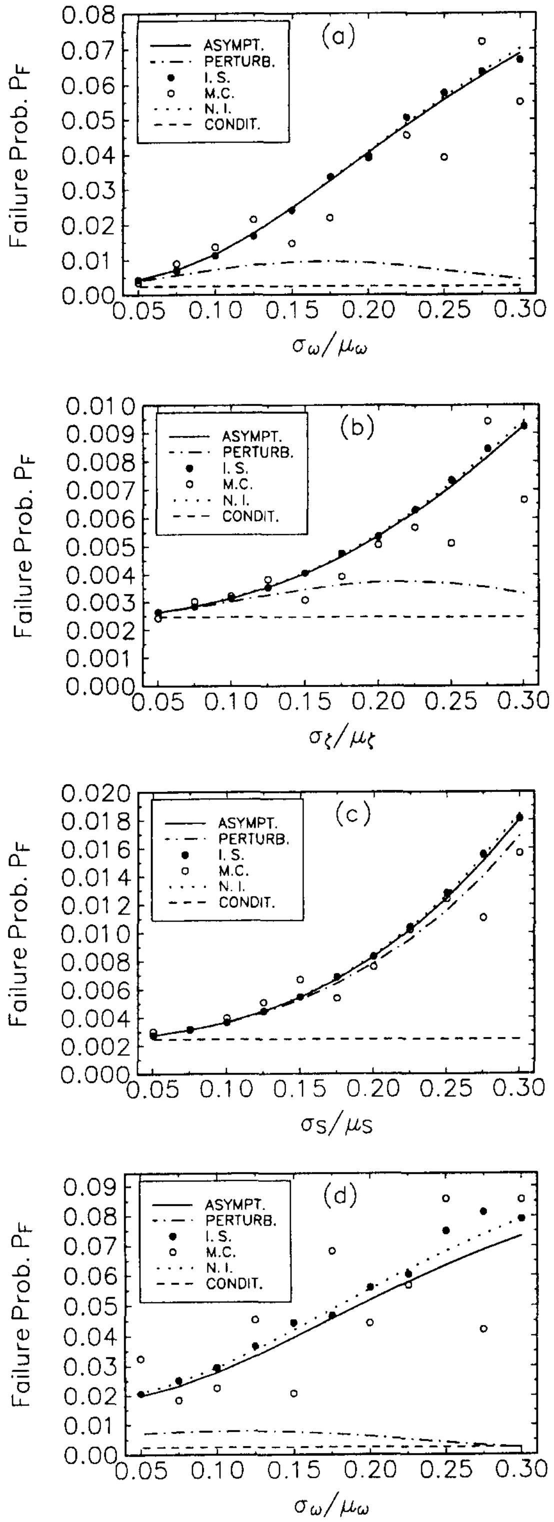

FIG. 2. Failure Probability P_F for Single DOF System (d = 3): (a) Uncertain ω; (b) Uncertain ζ; (c) Uncertain S; (d) Uncertain ω, ζ, and S

Again, the importance sampling method converges much faster than the Monte Carlo simulation.

Comparing the failure probabilities to the conditional failure $F(\mu_\theta)$ in Table 1 and Fig. 2, it is clear that uncertainties are very important, since they can change the failure probabilities by orders of magnitude. Also, comparing the values of the failure probabilities in Figs. 2(a), 2(b), and 2(c), one can conclude that the uncertainty in the frequency is much more important than the uncertainties in the damping and input power spectral density.

TABLE 1. Failure Probabilities ($\times 10^{-4}$) for Different Threshold Levels: One DOF System; Uncertain ω, ζ, and S

Threshold level × $\sqrt{2}\sigma_r(\mu_\theta)$ (1)	3.0 (2)	3.5 (3)	4.0 (4)
Conditional $[F(\mu_\theta)]$	26.65	0.96	0.023
Asymptotic	519.0	204.4	80.61
Perturbation	62.04	1.9	0.028
Importance sampling (Monte Carlo) 10^2 samples	527.2 (742.2)	206.6 (361.1)	86.53 (69.02)
Importance sampling (Monte Carlo) 10^3 samples	543.8 (621.9)	215.6 (258.9)	87.98 (75.00)
Importance sampling (Monte Carlo) 10^4 samples	552.8 (578.7)	216.7 (227.7)	86.01 (88.52)
Numerical integration	555.5	217.8	85.53

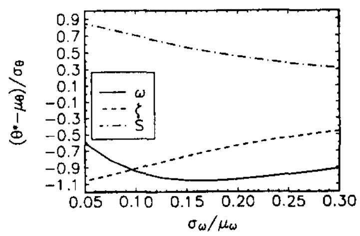

FIG. 3. Location of Maximum θ* of Integrand for P_F for Case of Fig. 2(d)

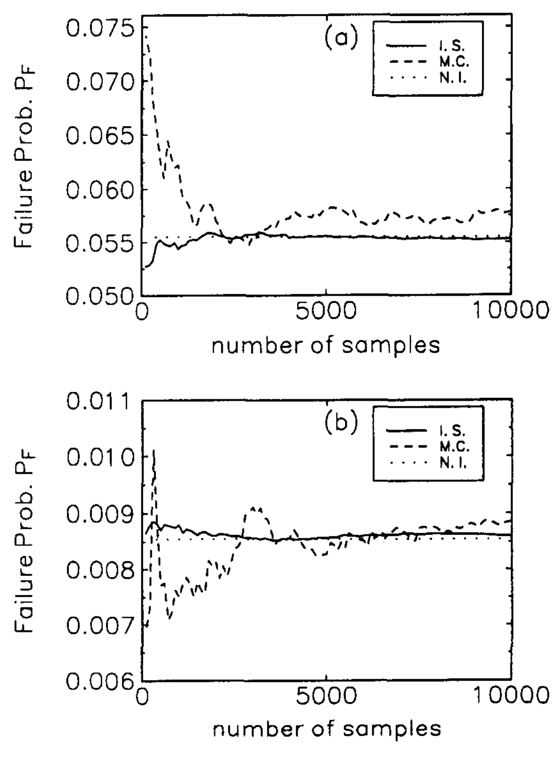

FIG. 4. Failure Probability Estimate $\tilde{P}_F$ for Single DOF System with Uncertain ω, ζ, and S: (a) d = 3; (b) d = 4

Uncertain Two Degree-of-Freedom Primary-Secondary System

The primary-secondary system is chosen because of the richness in its dynamic characteristics and the significant effects that uncertainties in the system parameters can have in its response and reliability (Igusa and Der Kiureghian 1988; Soong and Chen 1989). The mass ratio $\varepsilon = m_s/m_p$ and the frequency ratio $\alpha = \omega_s/\omega_p$ are important variables controlling the dynamics of the two DOF system shown schematically in Fig. 1(b). For illustrative purposes, the masses are assumed to be deterministic and the mass ratio is taken to be $\varepsilon = 0.01$ so that the system is representative of a two DOF primary-secondary structure. The uncertain parameters are chosen to be the natural frequencies ω_p and ω_s and the damping ratios ζ_p and ζ_s of the two oscillators. All uncertainties are modeled by lognormal variables that are assumed to be independent. The following most probable values are chosen for the system parameters: $\mu_{\zeta_p} = 5\%$, $\mu_{\zeta_s} = 2\%$, $\mu_{\omega_p} = 1$, and $\mu_{\omega_s} = \alpha\mu_{\omega_p}$. The ratios of the standard deviations to the most probable values are chosen to be 0.1 for the frequencies and 0.25 for the damping ratios.

The response quantity of interest is the restoring force per unit secondary mass, $r(t, \boldsymbol{\theta}) = \omega_s^2[x_s(t) - x_p(t)]$, of the spring connecting the secondary mass m_s to the primary mass m_p. Since $r(t, \boldsymbol{\theta})$ is a Gaussian stochastic process for a given $\boldsymbol{\theta}$, (24) and (25) can again be applied to compute the conditional probability of a stationary portion of the stochastic process $r(t, \boldsymbol{\theta})$ exceeding the level b for the first time. The quantities $\sigma_r^2(\boldsymbol{\theta})$ and $\sigma_{\dot{r}}^2(\boldsymbol{\theta})$ involved in (25) are the second-order moments of the response $r(t, \boldsymbol{\theta})$ and its time derivative, which can be obtained by solving the Lyapunov equation of the system for the covariance matrix of the response (Soong and Grigoriu 1993). For this particular case, analytical solutions involving the system parameters $\boldsymbol{\theta}$ can be written for the two moments.

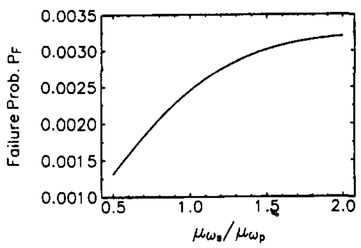

FIG. 5. Conditional Failure Probability $F(\mu_\theta)$ for Two DOF System ($d = 3$)

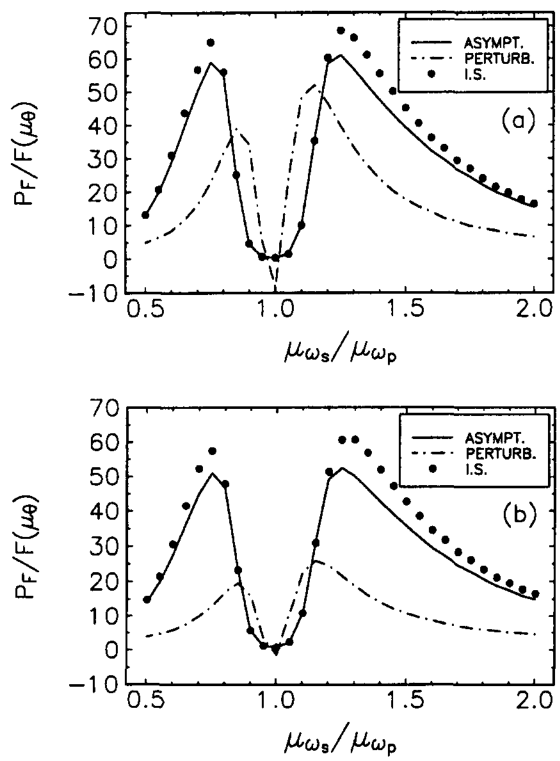

FIG. 6. Normalized Failure Probability $P_F/F(\mu_\theta)$ for Two DOF Primary-Secondary System: (a) Case A: Uncertain ω_p and ω_s; (b) Case B: Uncertain ω_p, ζ_p, ω_s, and ζ_s

The following three cases, namely, A, B, and C, will be considered. Case A corresponds to uncertainties in the frequency ratios ω_p and ω_s; case B corresponds to uncertainties in the frequencies and damping ratios ω_p, ω_s, ζ_p, and ζ_s; and case C corresponds to uncertainties in b, S, and T, as well as in ω_p, ω_s, ζ_p, and ζ_s. For case C, the uncertainties in b, S, and T are modeled by independent lognormal variables with most probable values $\mu_b = d\sqrt{2}\sigma_r(\boldsymbol{\mu}_\theta)$, $\mu_S = 1$ and $\mu_T = 10\ (2\pi/\mu_{\omega_s})$. The standard deviations are chosen as $\sigma_b = 0.1\mu_b$, $\sigma_S = 0.2\mu_S$, and $\sigma_T = 0.2\mu_T$.

The conditional probability of failure $F(\boldsymbol{\mu}_\theta)$ for the most probable values $\boldsymbol{\mu}_\theta$ of the system parameters is plotted in Fig. 5 for values of the frequency ratio $\alpha = \mu_{\omega_s}/\mu_{\omega_p}$ ranging from 0.5 to 2.0. Figs. 6(a) and 6(b) show the failure probabilities, normalized by the conditional failure probability $F(\boldsymbol{\mu}_\theta)$, for cases A and B, respectively. The results obtained from the asymptotic expansion are compared to those obtained from the perturbation method and the importance sampling technique computed for 10,000 samples. Since the accuracy of the asymptotic method for very small failure probabilities is not directly evident from Fig. 6, the failure probabilities are replotted in Fig. 7 using a logarithmic scale.

The asymptotic method performs very well in terms of predicting both the quantitative and qualitative features of the failure probabilities as a function of the frequency ratio $\mu_{\omega_s}/\mu_{\omega_p}$. In contrast, the perturbation method not only gives large errors, but also is unable to predict the qualitative features of the behavior of the failure probabilities. In particular, at the frequency ratio $\mu_{\omega_s}/\mu_{\omega_p} = 1.0$, it gives unrealistic negative values for the failure probability. The failure probabilities corresponding to case C are listed in Table 2 for three values of

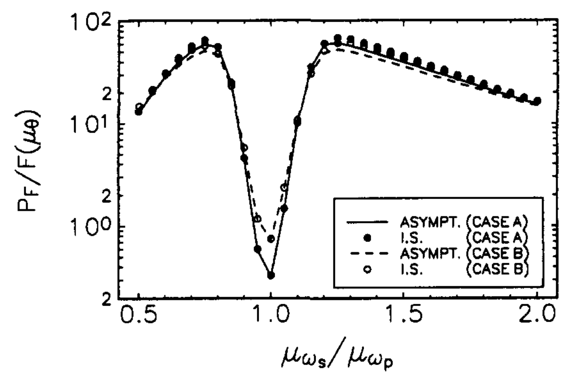

FIG. 7. Normalized Failure Probability $P_F/F(\mu_\theta)$ for Two DOF Primary-Secondary System for Cases A and B

TABLE 2. Failure Probabilities ($\times 10^{-4}$) for Different Threshold Levels; Two DOF System, Seven Uncertain Parameters, $\mu_{\omega_s}/\mu_{\omega_p} = 1.3$

Threshold level $\times \sqrt{2}\sigma_r(\boldsymbol{\mu}_\theta)$ (1)	3.0 (2)	3.5 (3)	4.0 (4)
Conditional [$F(\boldsymbol{\mu}_\theta)$]	28.52	1.108	0.026
Asymptotic	1,462	953.8	606.2
Perturbation	861.3	59.08	2.235
Importance sampling (Monte Carlo) 10^2 samples	2,015 (1,627)	1,268 (1,067)	776.2 (656.4)
Importance sampling (Monte Carlo) 10^3 samples	1,977 (1,734)	1,247 (1,087)	756.8 (647.4)
Importance sampling (Monte Carlo) 10^4 samples	1,875 (1,852)	1,178 (1,157)	712.7 (698.1)

the most probable threshold levels μ_b, and for the frequency ratio $\mu_{\omega_s}/\mu_{\omega_p} = 1.3$. The asymptotic expansion gives good results, while the perturbation method provides very poor results with errors of several orders of magnitude.

The location of the maximum $\boldsymbol{\theta}^*$ of the integrand for the probability of failure is shown in Fig. 8 as a function of the frequency ratio. The fact that $\boldsymbol{\theta}^*$ has moved further away from the most probable values $\boldsymbol{\mu}_\theta$ suggests that the importance sampling method can be advantageously used to accelerate the convergence of the simulation method. The failure probability estimate predicted by the two simulation methods as a function of the number of samples is shown in Fig. 9(a) for case B with $d = 3$, and in Fig. 9(b) for case C with $d = 4$. The coefficient of variation cov$[\tilde{P}_F]$ corresponding to the estimate $\tilde{P}_F$ in Figs. 9(a) and 9(b) is also computed, and the results are presented in Figs. 10(a) and 10(b), respectively. Again, the much faster convergence of the importance sampling method compared to the Monte Carlo simulation is evident in Figs. 9 and 10 in both cases.

Fig. 11 shows the conditional stationary second moment $\sigma_r^2(\boldsymbol{\mu}_\theta) = E[r^2|\boldsymbol{\mu}_\theta]$ of the secondary system spring force $r(t, \boldsymbol{\mu}_\theta)$, normalized by the conditional stationary moment computed at $\mu_{\omega_s}/\mu_{\omega_p} = 1.0$. The large amplification of the spring force variance at $\mu_{\omega_s}/\mu_{\omega_p}$ close to unity suggests that uncertainties in the structural parameters will be important in this region. The stationary second-moment $E[r^2]$, normalized by the conditional variance $\sigma_r^2(\boldsymbol{\mu}_\theta) = E[r^2|\boldsymbol{\mu}_\theta]$, is plotted in Figs. 12(a) and 12(b) for cases A and B, respectively. For comparison, results obtained from the second-order perturbation method and the importance sampling method using 1,000 samples for case A and 10,000 samples for case B are also included in these figures. Results from the more accurate numerical integration method are computed and shown only for case A in Fig. 12(a).

The second-order perturbation method gives poor results for the variance of the secondary-system spring force under conditions close to tuning, even if the level of uncertainties is relatively small. An example is case A presented in Fig. 12(a), which corresponds to relatively small levels of uncertainties since $\sigma_{\omega_p}/\mu_{\omega_p} = 0.1$ and $\sigma_{\omega_s}/\mu_{\omega_s} = 0.1$. When $\mu_{\omega_s}/\mu_{\omega_p} = 1$, the perturbation method gives unrealistic negative values for the variance. The asymptotic expansion gives better and more reliable estimates of the variance over the whole range of variation of the frequency ratio $\mu_{\omega_s}/\mu_{\omega_p}$.

The deviation of the results shown in Figs. 6, 7, and 12 from unity is directly related to the effect of the uncertainties on the failure probabilities and second-order moments. Values larger than one imply that the uncertainties are important, and neglecting them gives unconservative results. On the other hand, values smaller than one imply that neglecting the un-

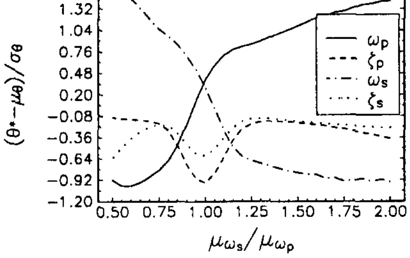

FIG. 8. Location of Maximum θ^* of Integrand for P_F for Case B of Fig. 6

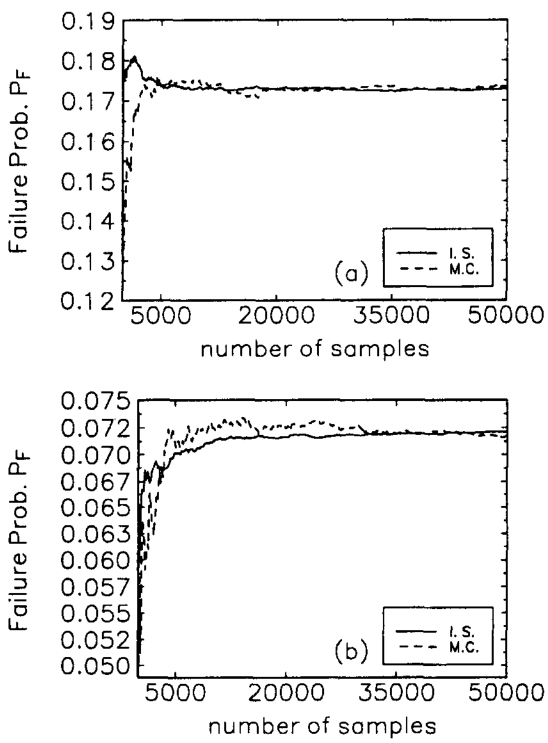

FIG. 9. Failure Probability Estimate $\tilde{P}_F$ for Two DOF Primary-Secondary System; $\omega_s/\omega_p = 1.3$: (a) Case B: Uncertain ω_p, ω_s, ζ_p, and ζ_s ($d = 3$); (b) Case C: Uncertain ω_p, ω_s, ζ_p, ζ_s, b, S, and T ($d = 4$)

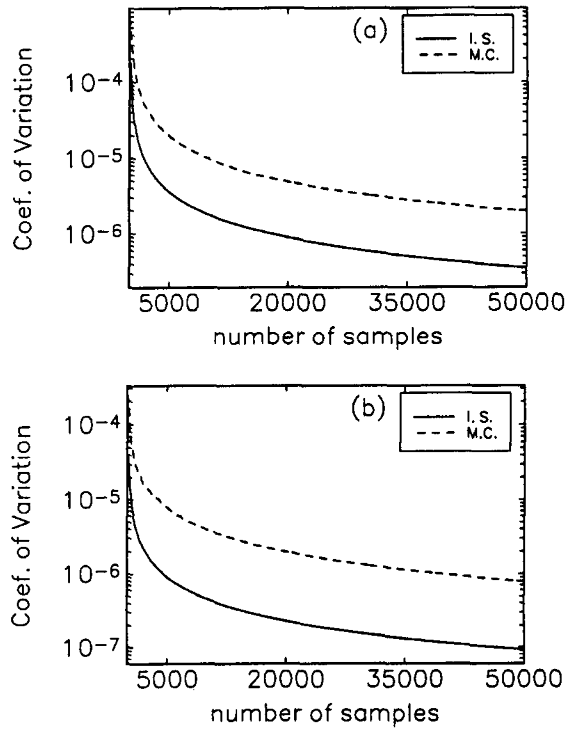

FIG. 10. Coefficient of Variation of Failure Probability Estimate $\tilde{P}_F$ for Two DOF Primary-Secondary System; $\omega_s/\omega_p = 1.3$: (a) Case B: Uncertain ω_p, ω_s, ζ_p, and ζ_s ($d = 3$); (b) Case C: Uncertain ω_p, ω_s, ζ_p, ζ_s, b, S, and T ($d = 4$)

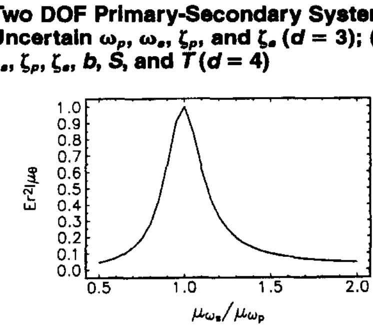

FIG. 11. Normalized Conditional Variance $E[r^2|\mu_\theta]$ of Secondary Spring Force for Two DOF Primary-Secondary System

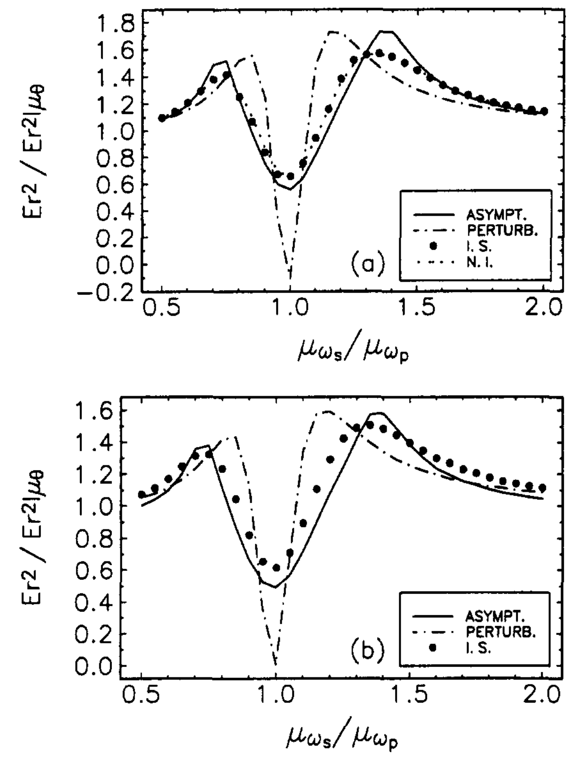

FIG. 12. Normalized Variance $E[r^2]$ for Secondary Spring Force for Two DOF Primary-Secondary System: (a) Case A: Uncertain ω_p and ω_s; (b) Case B: Uncertain ω_p, ω_s, ζ_p, and ζ_s

certainties will result in conservative values for the response. From the large values in Fig. 6, and the similarity of Figs. 6(a) and 6(b), it can be seen that the failure probabilities are very sensitive to uncertainties in the frequencies, but are less sensitive to uncertainties in the damping ratios for nearly but not perfectly tuned conditions. Neglecting the uncertainties in the frequencies will give highly unconservative results. The uncertainties in the damping ratios produce significant changes in the failure probability only when the frequency ratio $\mu_{\omega_s}/\mu_{\omega_p}$ is close to 1.0. Under perfectly tuned conditions, the failure probabilities are lower than that of the deterministic system, which means that analyses based on the most probable system will be conservative. Similar conclusions can be drawn for the moments of the secondary spring force, since the pattern of the moment variation presented in Fig. 12 is similar to the pattern shown in Fig. 6 for the variation of the failure probabilities.

Summarizing, the asymptotic method can be used to draw reliable qualitative conclusions about the behavior of uncertain linear dynamic systems for both the failure probabilities and the response moments, whereas this is not so if the results of the perturbation method are used. Also, the asymptotic method provides more reliable quantitative results; in particular, for failure probabilities, the perturbation method performs very poorly.

CONCLUSIONS

The primary focus of the present paper is to introduce a new asymptotic approximation for probability integrals of the type arising in the study of uncertain dynamic systems, and to examine the weaknesses and strengths of the asymptotic formulas with simple examples. The approach is used to derive simple analytical formulas for the probability of failure and response moments in uncertain dynamic systems, and the results are compared to some existing approximations.

When the asymptotic approximation is applied to compute the probability of failure, it is found to give the same result as a more involved asymptotic approximation, which first transforms the probability integral to a type of reliability integral over an unsafe domain determined by an artificial limit state function and then applies an asymptotic approximation available for integrals of this type. In the cases examined, the new method is generally much more accurate than the second-order perturbation method, which can be completely unreliable, especially when used to predict failure probabilities. When deriving "exact" results to check the accuracy of the new method, it is much more efficient to use an importance sampling method than the Monte Carlo simulation.

The procedure described in this paper is general and can be used to obtain the reliabilities and moments of multi-DOF linear and nonlinear systems subjected to nonwhite Gaussian and non-Gaussian excitation processes. The conditional quantities required in the present method can be obtained from random vibration theory available for each specific case.

ACKNOWLEDGMENTS

This paper is based upon work partly supported by the National Science Foundation under grant BCS-9309149 and by the Hong Kong UPGC Research Infrastructure grant, RIG 94/95.EG02. This support is gratefully acknowledged. This paper was written while J. L. Beck was on sabbatical leave at the Hong Kong University of Science and Technology, and the kind hospitality of J. C. Chen, director of the Research Center, is greatly appreciated. The writers also thank Siu-Kui Au, graduate student at the Hong Kong University of Science and Technology, for his assistance in carrying out part of the numerical computations.

APPENDIX I. SIMPLIFICATIONS OF EXISTING SORM-BASED RELIABILITY RESULT

The probability of failure given by (16) can also be rewritten in the form (Cherng and Wen 1994)

$$P_F = \int_{g(\boldsymbol{\theta},y)<0} p_y(y)p(\boldsymbol{\theta})\, d\boldsymbol{\theta}\, dy \quad (27)$$

where $g(\boldsymbol{\theta}, y) = \text{CDF}^{-1}[1 - F(\boldsymbol{\theta})] - y$ = a limit state function; y = a new nonnegative variable with probability density function $p_y(y)$; and CDF denotes the cumulative density function of y. Choosing $p_y(y) = e^{-y}$, the limit state function can be written in the form

$$g(\mathbf{x}) = -\ln[F(x_1, \ldots, x_n)] - x_{n+1} \quad (28)$$

where $\mathbf{x}^T = [\boldsymbol{\theta}^T, y]$. This function is then approximated by a second-order surface at the point where failure is most likely to occur (Breitung 1989). This point $\mathbf{x}^*$ is obtained by maximizing the log-likelihood function

$$\hat{l}(\mathbf{x}) = \ln p(x_1, \cdots, x_n) - x_{n+1} \quad (29)$$

subject to the constraint $g(\mathbf{x}) = 0$. Using (28), the determination of $\mathbf{x}^*$ reduces to solving the unconditional optimization problem for

$$l(x_1, \cdots, x_n) = \ln p(x_1, \cdots, x_n) + \ln F(x_1, \cdots, x_n) \quad (30)$$

Using the asymptotic approximation of Breitung (1989) for integrals of the form (27), an approximation for the probability of failure is obtained

$$P_F \sim (2\pi)^{n/2} \exp(-\beta_0^2) \frac{1}{\sqrt{|J(\mathbf{x}^*)|}} \quad (31)$$

where

$$\beta_0 = \sqrt{-\hat{l}(\mathbf{x}^*)} \quad (32)$$

$$J(\mathbf{x}^*) = [\nabla\hat{l}(\mathbf{x}^*)]^T \mathbf{C}(\mathbf{x}^*)\nabla\hat{l}(\mathbf{x}^*) \quad (33)$$

in which $\nabla\hat{l}(\mathbf{x}^*)$ = gradient vector of $\hat{l}(\mathbf{x})$ evaluated at $\mathbf{x} = \mathbf{x}^*$;

(i, j) element of matrix $\mathbf{C}(\mathbf{x}) = (i, j)$ cofactor of a matrix $\hat{\mathbf{H}}(\mathbf{x})$; and the (i, j) element of $\hat{\mathbf{H}}(\mathbf{x})$ is

$$\hat{H}_{ij}(\mathbf{x}^*) = \frac{\partial^2 \hat{l}(\mathbf{x}^*)}{\partial x_i \partial x_j} - \frac{|\nabla \hat{l}(\mathbf{x}^*)|}{|\nabla g(\mathbf{x}^*)|} \frac{\partial^2 g(\mathbf{x}^*)}{\partial x_i \partial x_j}, \quad i, j = 1, \cdots, n + 1 \tag{34}$$

This result for the probability of failure has been reported by Cherng and Wen (1994) for the case of independent $x_1, \cdots, x_n$, i.e., using $\ln p(x_1, \cdots, x_n) = \ln p_{x_1}(x_1) + \cdots + \ln p_{x_n}(x_n)$ in (30).

One can proceed further, however, to simplify considerably the result given before. Note that at the point $\mathbf{x}^*$ corresponding to the maximum of $l(x_1, \cdots, x_n)$ given by (30) the following equations are satisfied:

$$\frac{1}{p(\mathbf{x}^*)} \frac{\partial p(\mathbf{x}^*)}{\partial x_i} = -\frac{1}{F(\mathbf{x}^*)} \frac{\partial F(\mathbf{x}^*)}{\partial x_i}, \quad i = 1, \cdots, n \tag{35}$$

Also note that $x_i^* = \theta_i^*$, $i = 1, \cdots, n$, where θ_i^* corresponds to the maximum of the function (3). The quantities $\nabla \hat{l}(\mathbf{x})$ and $\nabla g(\mathbf{x})$ take the form

$$[\nabla \hat{l}(\mathbf{x})]^T = \left[\frac{1}{p(\mathbf{x})} \frac{\partial p(\mathbf{x})}{\partial x_1}, \cdots, \frac{1}{p(\mathbf{x})} \frac{\partial p(\mathbf{x})}{\partial x_n}, -1\right] \tag{36}$$

$$[\nabla g(\mathbf{x})]^T = \left[\frac{1}{F(\mathbf{x})} \frac{\partial F(\mathbf{x})}{\partial x_1}, \cdots, \frac{1}{F(\mathbf{x})} \frac{\partial F(\mathbf{x})}{\partial x_n}, -1\right] \tag{37}$$

Evaluating (36) and (37) at the point $\mathbf{x}^*$ and making use of (35) gives

$$|\nabla \hat{l}(\mathbf{x}^*)| = |\nabla g(\mathbf{x}^*)| \tag{38}$$

which can be used to simplify (34) to

$$\hat{H}_{ij}(\mathbf{x}^*) = \frac{\partial^2 \hat{l}(\mathbf{x}^*)}{\partial x_i \partial x_j} - \frac{\partial^2 g(\mathbf{x}^*)}{\partial x_i \partial x_j}, \quad i, j = 1, \cdots, n + 1 \tag{39}$$

Using (28) and (29), it is straightforward to show that

$$\hat{H}_{i(n+1)}(\mathbf{x}) = \hat{H}_{(n+1)i}(\mathbf{x}) = 0, \quad i = 1, \cdots, n + 1 \tag{40}$$

This implies that all but the $(n + 1, n + 1)$ elements of $\mathbf{C}(\mathbf{x})$ are equal to zero. The nonzero element is $C_{(n+1)(n+1)}(\mathbf{x}) = \det[\mathbf{H}(\mathbf{x}^*)]$, where $\mathbf{H}(\mathbf{x}^*)$ is an $n \times n$ matrix with elements

$$H_{ij}(\mathbf{x}^*) = \frac{\partial^2 \hat{l}(\mathbf{x}^*)}{\partial x_i \partial x_j} - \frac{\partial^2 g(\mathbf{x}^*)}{\partial x_i \partial x_j} = \frac{\partial^2 l(\mathbf{x}^*)}{\partial x_i \partial x_j}, \quad i, j = 1, \cdots, n \tag{41}$$

Substituting (36) and (37) into (33), and considering the simplified form of $\mathbf{C}(\mathbf{x})$, yields

$$J(\mathbf{x}^*) = \det[\mathbf{H}(\mathbf{x}^*)] \tag{42}$$

which is the same as the $\det[\mathbf{L}(\boldsymbol{\theta}^*)]$ appearing in (8), where $\mathbf{L}(\boldsymbol{\theta}^*)$ is given by (6). Finally, noting that $\exp(-\beta_0^2) = \exp[\hat{l}(\mathbf{x}^*)] = \exp[l(\boldsymbol{\theta}^*)] = F(\boldsymbol{\theta}^*)\, p(\boldsymbol{\theta}^*)$, approximation (31) is the same as (8) derived in this paper where an asymptotic approximation is applied on the original failure probability integral (16).

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APPENDIX III. NOTATION

The following symbols are used in this paper:

- b = threshold level;
- $\mathbf{C}$ = cofactors of $\hat{\mathbf{H}}$;
- cov = coefficient of variation of random variable;
- d = normalized threshold level;
- E = expected value;
- F = conditional failure probability;
- g = limit state function;
- $\mathbf{H}, \hat{\mathbf{H}}, \mathbf{L}, \mathbf{T}$ = Hessian matrices;
- $H_{ij}, \hat{H}_{ij}, L_{ij}, T_{ij}$ = (i, j) component of Hessian matrices;
- h = function in (1);
- I = value of integral;
- J = function defined in (33);
- l = function defined in (3);
- $\hat{l}$ = log-likelihood function;
- P_F = overall failure probability;
- $\tilde{P}_F$ = overall failure probability estimate;
- P_R = overall reliability;
- p = probability density function;

$p_y(y)$ = density function of y;
R = conditional reliability;
$\mathfrak{R}^n$ = n-dimensional real vector space;
r = system response;
$\dot{r}$ = system response derivative;
S = excitation spectral density;
$s(\delta)$ = sensitivity function;
T = period of oscillations;
V_{ij} = (i, j) element of covariance of $\boldsymbol{\theta}$;
var = variance of random variable;
w = importance sampling density;
$\mathbf{x}$ = random vector;
y = random variable;
$\mathbf{z}$ = state response vector;
β_0 = parameter defined in (32);
δ = system parameter;
$\partial/\partial\delta$ = partial derivative;
ζ = damping ratio;
η = mth moment of $\mathbf{z}$;
Θ = subregion of $\mathfrak{R}^n$;
$\boldsymbol{\theta}$ = uncertain parameters;
$\bar{\boldsymbol{\theta}}$ = mean value of $\boldsymbol{\theta}$;
θ_i = ith component of $\boldsymbol{\theta}$;
$\boldsymbol{\theta}^*$ = maximum point of $h(\boldsymbol{\theta})\, p(\boldsymbol{\theta})$;
κ = hp/w;
λ_i = ith eigenvalue of $\mathbf{L}$;
μ_y = mean value of random variable y;
ν = expected rate of up-crossing a level;
σ_y = standard deviation of random variable y;
ω = natural frequency; and
∇ = gradient symbol.

Subscripts

p = primary system; and
s = secondary system.